SADDLEPOINT APPROXIMATIONS FOR CREDIT PORTFOLIO DISTRIBUTIONS WITH APPLICATIONS IN EQUITY RISK MANAGEMENT

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Abstract. We study saddlepoint approximations to the tail-distribution for credit portfolio losses in continuous time intensity based models under conditional independent homogeneous settings. In such models, conditional on the filtration generated by the individual default intensity up to time t, the conditional number of defaults distribution (in the portfolio) will be a binomial distribution that is a function of a factor Z_t which typically is the integrated default intensity up to time t. This will lead to an explicit closed-form solution of the saddlepoint equation for each point used in the number of defaults distribution when conditioning on the factor Z_t, and we hence do not have to solve the saddlepoint equation numerically. The ordo-complexity of our algorithm computing the whole distribution for the number of defaults will be linear in the portfolio size, which is a dramatic improvement compared to e.g. recursive methods which have a quadratic ordo-complexity in the portfolio size. The individual default intensities can be arbitrary as long as they are conditionally independent given the factor Z_t in a homogeneous portfolio. We also outline how our method for computing the number of defaults distribution can be extend to heterogeneous portfolios. Furthermore, we show that all our results can be extended to hold for any factor copula model. We give several numerical applications and in particular, in a setting where the individual default intensities follow a CIR process we study both the tail distribution and the number of defaults distribution. We then repeat similar numerical studies in a one-factor Gaussian copula model. We also numerically benchmark our saddlepoint method to other computational methods. Finally, we apply of our saddlepoint method to efficiently investigate Value-at-Risk for equity portfolios where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities driven by a one-factor Gaussian copula model were we focus on Value-at-Risk as function of the default correlation parameter in the one-factor Gaussian copula model.

Keywords: credit portfolio risk; intensity-based models; factor models; credit copula models; Value-at-Risk; conditional independent dependence modelling; saddlepoint-methods; Fourier-transform methods; numerical methods; equity portfolio risk; stock price modelling with jumps;

JEL Classification: G33; G13; C02; C63; G32 .

1. Introduction

Consider a credit portfolio consisting of equally weighted obligors and let N_t be the number of defaults in this portfolio up to time t. Finding the "number of defaults" distributions N_t for a credit portfolio in continuous time is at the core of credit portfolio risk management, portfolio OTC-loss computations, portfolio counterparty risk management, hybrid equity-credit risk management (partly studied in this paper) and for computing various quantities needed by central counterparties. Besides this, the distributions for N_t are also needed when pricing portfolio credit derivatives.

There exists a variety of different algorithms and techniques to find the distributions for N_t. In this paper we study saddlepoint approximations for the numboer of default distribution in continuous time intensity based credit portfolios. We focus on homogeneous conditional independent credit portfolio models where the default intensities $\lambda_t = \lambda(X_t)$ are the same for all obligors and X_t is some stochastic process. If $\mathcal{F}_t^X$ is the filtration generated by $\lambda_t = \lambda(X_t)$ then conditional on $\mathcal{F}_\infty^X$ and for a fixed time t, the random variable N_t will be a conditional binomial distribution. More specific, if one defines the "factor" Z_t as $Z_t = \int_0^t \lambda_s \, ds$, then conditional on $\mathcal{F}_\infty^X$ we will have a conditional binomial distribution which will be an explicit function of the factor Z_t. By using saddlepoint theory for a binomial distribution which is a sum of iid Bernoulli random variables, this will lead to an explicit closed-form solution of the saddlepoint equation for each point x for the conditional number of defaults distribution $\mathbb{P}\left[N_t \geq x \,|\, \mathcal{F}_\infty^X \right]$ and we thus do not have to solve the saddlepoint equation numerically for each outcome of the factor Z_t. The individual default intensities can be arbitrary as long as they are conditionally independent given the

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factor Z_t in a homogeneous portfolio. We are primary considering exchangeable credit portfolios, but we will also give a detailed description on how to deal with heterogeneous credit portfolios.

Furthermore, we show that all our results also hold when N_t is generated by default times coming from any type of factor copula model, including the widely used one-factor Gaussian copula model and the Clayton copula model.

In the numerical section of the paper, we give several numerical applications and in particular, in a setting where the individual default intensities follow a CIR process we study the time evolution of the distributions $\mathbb{P}[N_t \geq x]$ and $\mathbb{P}[N_t = k]$. We then repeat similar numerical studies in a one-factor Gaussian copula model. There exists a huge amount of applications where the distributions $\mathbb{P}[N_t \geq x]$ and $\mathbb{P}[N_t = k]$ are used, particular in credit risk, for example risk management of credit portfolios done under the real probability measure, but also for credit portfolio derivative pricing. However, in this paper we will in Section 7 focus on applications of $\mathbb{P}[N_t = k]$ in equity risk management in a stock price model developed in Herbertsson (2023*a*) where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities, for example corporates or sovereign states. By "exogenous" we here mean that the entities, for example companies, will not be represented in the stock portfolio, that is stocks issued by the defaultable corporates are not present in the stock portfolio in our studies. The default times can come from any type of credit portfolio model. In this paper we will perform some complementary numerical studies of the stock price model developed in Herbertsson (2023*a*), which are not present in Herbertsson (2023*a*). More specific, in Section 10 we will present numerical results to Value-at-Risk for a large stock portfolio, as function of the default correlation parameter in the one-factor Gaussian copula model, at different timepoints and for different confidence levels. Such studies are not done in Herbertsson (2023*a*), and are directly dependent on efficient and fast computations of the distribution $\mathbb{P}[N_t = k]$.

The saddlepoint approach applied to conditional distributions for portfolio credit risk have been studied previously in the literature. For example, conditional versions of the saddlepoint approach in credit risk with applications to the Vasciek portfolio credit loss model in static settings have been studied in e.g. Huang, Oosterlee & van der Weide (2007) and Martin (2011), see for Section 2.8-2.10 on pp.548-550 in Martin (2011) and pp. 100-102 in Huang et al. (2007). Gordy (2002) applies the conditional saddlepoint technique to the Creditrisk+-model in static time. For more about the conditional versions of the saddlepoint approach in credit risk, see e.g. Chapter 5 in Kwok & Zheng (2018), in particular pp.114-115 in Kwok & Zheng (2018). The approach in Gordy (2002), Huang et al. (2007) and Martin (2011) uses heterogeneous portfolios, which forces one to solve the conditional saddlepoint equation numerically and where the conditional factor is a static random variable which typically has normal distribution. For example, in Section 2.9 on p.549 in Martin (2011) the author writes that a large portion of the computational time is spent on finding the solution of the conditional saddlepoint equation which has to be done for each value of the factor.

We will use a different approach from several different point of views, compared to e.g. Gordy (2002), Huang et al. (2007) and Martin (2011). First, we will primary consider an intensity based setting which means that the factor Z_t will be an integrated intensity, but we will also prove that our the method works for traditional factor models, as studied in e.g. Gordy (2002), Gregory & Laurent (2005), Huang et al. (2007) and Martin (2011). Secondly, we will mainly consider homogeneous portfolios but this assumption will later be relaxed to consider heterogenous portfolios that constitute of several homogeneous subportfolios, similar to the ideas presented in Papageorgiou & Sircar (2009). The fact that we are primary considering homogeneous portfolios implies that we will for each fixed t have a conditional binomial distribution which will lead to a explicit closed form solution of the saddlepoint equation for each point x given the factor Z_t, and we thus do not have to solve the saddlepoint equation for each point x given the factor Z_t which can be very time consuming for heterogeneous portfolio. Thirdly, since the model is in continuous time we can calibrate the individual default probabilities for the default times (which will be same for all obligors) to e.g. a term-structure of these default probabilities and then use the parameters in the model to compute the loss-distributions for arbitrary values of t, knowing that the model have been consistently calibrated to available data of e.g. the term-structure of the individual default probability.

Another major difference between our approach and the ones in e.g Gordy (2002), Huang et al. (2007) and Martin (2011), is that we in the intensity based case use the saddlepoint in combination with Fourier inversion methods, alteratively we can use the saddlepoint method twice. First, the saddlepoint approach is used to expand the conditional binomial distribution in closed formulas without needing to solve the saddlepoint equation numerically for this conditional distribution. Then we use the either Fourier inversion

methods or the saddlepoint approach to find good approximations to the density $f_{Z_t}(z)$ to the random variable Z_t which is possible due to the fact that we often have highly analytical expression for the moment generating function to the random variable $Z_t = \int_0^t \lambda(X_s)\, ds$, in terms of the parameters describing the individual default intensity $\lambda(X_t)$.

To this end, we also remark that there exists recursive algorithms for computing $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$, see e.g. in Andersen, Sidenius & Basu (2003) or Andersen & Sidenius (2004), which works both for heterogeneous and homogeneous portfolios. However, for a portfolio with m obligors, the recursive algorithm introduced in Andersen et al. (2003) implies that computing the whole distribution $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$, that is for all $k = 0, 1, \ldots, m$, is of order $O(m^2)$, see for example on p.68 in Andersen et al. (2003) where the authors write *"the cost of building the loss distribution grows as roughly the square of the basket size"*. Another drawback with the recursive method is that it will not give any explicit formulas for $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right]$ or $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$.

The saddlepoint approach introduced in this paper will produce a method of order $O(1)$ when computing $\mathbb{P}\left[N_t \geq x \,|\, \mathcal{F}_\infty^X\right]$ and thus also $O(1)$ for computing $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$. Hence, for a portfolio with m obligors, our approach will therefore for computing the whole distribution $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$, that is for all $k = 0, 1, \ldots, m$, be of order $O(m)$, i.e. linear in the portfolio size m, which is an dramatic improvement compared with the quadratic, $O(m^2)$, recursive algorithm introduced in Andersen et al. (2003). Another advantage with our saddlepoint approach is that it leads to explicit formulas for $\mathbb{P}\left[N_t \geq x \,|\, \mathcal{F}_\infty^X\right]$ and thus also for $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$ in terms of x, k, m and the $\mathcal{F}_\infty^X$-conditional individual default probability. As pointed out above, our saddlepoint method will also work for conditional factor models, such as discussed in e.g Gordy (2002), Gregory & Laurent (2005), Huang et al. (2007) and Martin (2011), and then we simply replace $\mathbb{P}\left[N_t \geq x \,|\, \mathcal{F}_\infty^X\right]$ and $\mathbb{P}\left[N_t = k \,|\, \mathcal{F}_\infty^X\right]$ with $\mathbb{P}\left[N_t \geq x \,|\, Z\right]$ and $\mathbb{P}\left[N_t = k \,|\, Z\right]$ for some static factor Z, and this will be discussed in Section 5 .

Besides the recursive algorithms discussed in e.g. Andersen et al. (2003) and Andersen & Sidenius (2004), there also exists Fourier methods where the characteristic function for the loss distribution is directly derived and used with Fourier inversion methods such as the FFT method and such techniques are discussed in e.g. Gregory & Laurent (2003) and Gregory & Laurent (2005). However, as pointed out in Andersen et al. (2003), such Fourier inversion methods seem to be much slower than recursive algorithms developed in Andersen et al. (2003) and Andersen & Sidenius (2004). For example, on p.68 in Andersen et al. (2003) the authors remark that they conducted numerical examples where the Fourier method could be up to 25 times slower than the recursive algorithm. Given that our saddlepoint approach has linear complexity in terms of the portfolio size, compared to the quadratic complexity of the recursive algorithms in Andersen et al. (2003), we therefore conclude that our method should also be much faster than the Fourier inversion methods proposed in Gregory & Laurent (2003) and Gregory & Laurent (2005).

The rest of the paper is organized as follows. First, in Section 2 we introduce notation and concepts that will be used in the rest of the paper. In Section 3 we outline so called intensity based models via the Cox-approach, which are conditional independent models, and also state some important results which will be the main feature allowing us to use the saddlepoint approximation method. Next, in Section 4 we first recap the saddlepoint approach and then use it to find very sharp approximations to the conditional tail-probabilities for the number of defaults distributions in an intensity based credit portfolio model as outlined in Section 3. Section 5 discuss how the results in Section 4 also will hold for factor copula models, such as the Gaussian one-factor model and the the Clayton copula. In Section 6 we show how our methods for the number of default distributions in homogeneous portfolios, can be relaxed to consider heterogenous portfolios that constitute of several homogeneous subportfolios, both for intensity based models and factor copula models. Section 7 focus on applications of $\mathbb{P}\left[N_t = k\right]$ in equity risk management in a stock price model developed in Herbertsson (2023*a*) where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities, for example corporates or sovereign states. By "exogenous" we here mean that the entities, for example companies, will not be represented in the stock portfolio, that is stocks issued by the defaultable corporates are not present in the stock portfolio in our studies. In Section 8 we study the time evolution of the distributions $\mathbb{P}\left[N_t \geq x\right]$ and $\mathbb{P}\left[N_t = k\right]$ when the individual default intensities follow a CIR process and in Section 9 we repeat similar numerical studies in a one-factor Gaussian copula model. In Section 10 we perform some complementary numerical studies of the stock price model briefly discussed in Section 7, and which are not present in Herbertsson (2023*a*). For example, in Section 10 we give numerical results to Value-at-Risk for a large stock portfolio, as function of the default correlation parameter in the one-factor Gaussian copula model, at different

timepoints and for different confidence levels. Such studies are not done in Herbertsson (2023*a*), and are directly dependent on efficient and fast computations of the distribution $\mathbb{P}[N_t = k]$. Finally, in Section 11 we numerically benchmark our method to another computational method, and show that the saddlepoint method is remarkable robust, while other methods will often fail numerically.

2. The credit portfolio loss distribution for homogeneous settings in general continuous time models.

Consider a homogeneous credit portfolio consisting of m equally weighted obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$. The number of defaults in the portfolio up to time t, denoted by $N_t^{(m)}$ is defined as

$$N_t^{(m)} = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}. \tag{2.1}$$

Throughout this paper we will assume that the default times $\tau_1, \tau_2 \ldots, \tau_m$ are generated from some conditional independent model, either via the intensity based approach or via a factor copula model. Furthermore, we are primary considering exchangeable credit portfolios, but in a second stage we will outline how to deal with heterogeneous portfolios in Section 6.

The main goal of this paper is to find compact and computational tractable approximations of the tail distributions $\mathbb{P}\left[N_t^{(m)} \geq x\right]$ for arbitrary sizes of m, i.e. m can be small or large. The technique that we will use is the celebrated saddlepoint approximation method but in a way that has not been used in the credit and finance literature before. The main idea in this paper is that when we in the above models conditional on some "factor", then $N_t^{(m)}$ will for a fixed timepoint t be a mixed binomial random variable, or equivalently, conditional on the factor then $\{1_{\{\tau_i \leq t\}}\}_{i=1}^m$ is an i.i.d Bernoulli sequence with some random probability. We can then use a conditional version of the Lugannani-Rice formula for the binomial distributions which will be fully analytic and where the saddlepoint equation is obtained in closed formula so that no numerical solution of this equation is needed. We then apply this method to find sharp approximations to the conditional tail distribution of $N_t^{(m)}$ and our approach works for arbitrary portfolio sizes m, where m can be small or large.

We will first and foremost focus on conditionally independent intensity based models, that is where the default times $\tau_1, \tau_2 \ldots, \tau_m$ have intensities with respect to some filtration, as will be discussed in Section 3 below, and in this settings sharp formulas for $\mathbb{P}\left[N_t^{(m)} \geq x\right]$ will be derived in Subsection 4.3. Later, we will in Section 5 extend the results from the intensity based setting in Subsection 4.3 and show that all the results in Subsection 4.3 also hold for any factor copula model, including the widely used one-factor Gaussian copula model and the Clayton copula model.

3. Conditional independent intensity based models and their default distributions in homogeneous settings

In this section we will outline so called intensity based models via the Cox-approach which are conditional independent models. We also state some important results for these models and their consequences, in particular for credit portfolios, which are a main feature that will be used in our saddle point approximations.

3.1. Construction of default times in the Cox-setting.

Let $(X_t)_{t\geq 0}$ be a d-dimensional stochastic process, i.e. $X_t = (X_{t,1}, X_{t,2}, \ldots, X_{t,d})$ in some space which typically is $\mathbb{R}^d$ where d is an integer, and let $\mathcal{F}_t^X = \sigma(X_s; s \leq t)$ be the filtration generated by the factor process X.

Consider m obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and let the mappings $\lambda_1, \lambda_2 \ldots, \lambda_m$ be the corresponding $\mathcal{F}_t^X$ default intensities, where $\lambda_i : \mathbb{R}^d \mapsto \mathbb{R}^+$ for each obligor i. This means that each default time τ_i is modeled as the first jump of a Cox-process, with intensity $\lambda_i(X_t)$. It is well known (see e.g. Lando (1998)) that given an i.i.d sequence $\{E_i\}$ where E_i is exponentially distributed with parameter one, such that all $\{E_i\}$ are independent of $\mathcal{F}_\infty^X$, then

$$\tau_i = \inf\left\{t > 0 : \int_0^t \lambda_i(X_s)ds \geq E_i\right\}. \tag{3.1.1}$$

Hence, for any $T \geq t$ we have

$$\mathbb{P}\left[\tau_i > t \,|\, \mathcal{F}_T^X\right] = \exp\left(-\int_0^t \lambda_i(X_s)ds\right) \tag{3.1.2}$$

and thus

$$\mathbb{P}\left[\tau_i > t\right] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_i(X_s)ds\right)\right]. \tag{3.1.3}$$

Note that the default times are conditionally independent, given $\mathcal{F}_\infty^X$. There are many examples of intensities which yields closed-form expressions, both for for (3.1.2) and (3.1.3), for example, when the intensity follows a CIR-process as in Bielecki, Cousin, Crépey & Herbertsson (2014*c*), or a shot-noise process as in Herbertsson, Jang & Schmidt (2011) or a finite-state Markov chain as in e.g. Herbertsson & Frey (2014). Below we give an example of the standard CIR-process.

3.1.1. *Example: intensity is a CIR-process.* Let $\lambda_t = \lambda\left(X_t\right) = X_t$ be a Cox-Ingersoll-Ross process (CIR-process). This means that

$$d\lambda_t = a\left(\mu - \lambda_t\right)dt + \sigma\sqrt{\lambda_t}dW_t \tag{3.1.1.1}$$

where W_t is a Brownian motion under the risk-neutral measures $\mathbb{P}$. Furthermore, one can then show that, see e.g. Subsection 9.5.2, pp.423-424 in McNeil, Frey & Embrechts (2005) or Appendix E, pp.292-293 in Lando (2004)

$$\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right)\,\middle|\,\mathcal{F}_t^X\right] = \exp\left(A(T-t) - B(T-t)\lambda_t\right) \tag{3.1.1.2}$$

where

$$A(s) = \frac{2a\mu}{\sigma^2}\ln\left(\frac{2\gamma e^{(\gamma+a)s/2}}{(\gamma+a)(e^{\gamma s}-1)+2\gamma}\right) \tag{3.1.1.3}$$

$$\gamma = \sqrt{a^2+2\sigma^2} \tag{3.1.1.4}$$

$$B(s) = \frac{2\left(e^{\gamma s}-1\right)}{(\gamma+a)(e^{\gamma s}-1)+2\gamma} \tag{3.1.1.5}$$

so $\mathbb{P}\left[\tau > T\right] = \exp\left(A(T) - B(T)\lambda_0\right)$.

3.2. **The credit portfolio loss distribution for homogeneous intensity based settings.** In this section we give a general discussion of how to find the distribution of the number of defaults in the portfolio up to time t, denoted by $N_t^{(m)}$ and defined as in Equation (2.1), that is $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, when the default times $\tau_1, \tau_2 \ldots, \tau_m$ are constructed as in (3.1.1) for a homogeneous portfolio. Thus, we assume that $\lambda_i(X_t) = \lambda(X_t)$ for all obligors in the portfolio. Later, this will be relaxed in Section 6.

Most of the previous credit literature for homogeneous credit portfolios have mainly been focused on so called large portfolio approximations of $N_t^{(m)}$ which only works accurately for very large m. In this paper we make no restriction on the portfolio size m, that is, we work with arbitrary sizes of m where m can be small or large. As we will see in Subsection 4.3, the numerical errors for the saddlepoint approximation to the distribution $\mathbb{P}\left[N_t^{(m)} \geq x\right]$ will be of the order $O(m^{-3/2})$ which thus quickly decreases when m increases. In practice however, as will be seen in our numerical studies in this paper, the numerical errors will be much smaller than $m^{-3/2}$ even for small values of the portfolio size m. This is important, since one sometimes also want to consider moderate sizes of m, in particular when working with heterogeneous portfolios that will be split into several different homogeneous credit portfolios that are merged together, as will be discussed in Section 6.

We first state the following proposition, which is more or less known in the credit literature, in order to introduce notation and concepts that will be used in the rest of the paper.

Proposition 3.1. *Consider a homogeneous credit portfolio with m entities with individual default intensities $\lambda_i(X_t) = \lambda(X_t)$ and default times τ_i generated via* (3.1.1)*. Define Z_t and $p(Z_t)$ as*

$$Z_t = \int_0^t \lambda(X_u)du \quad \text{and} \quad p(Z_t) = 1 - e^{-Z_t}. \tag{3.2.1}$$

Then,

$$\mathbb{P}\left[\tau_i \leq t \,|\, \mathcal{F}_\infty^X\right] = p(Z_t) \tag{3.2.2}$$

and $\tau_1, \dots, \tau_m$ *are conditional independent given* $\mathcal{F}_\infty^X$ *where*

$$\lim_{m\to\infty} \frac{N_t^{(m)}}{m} = p(Z_t) \quad \textit{a.s. under the random measure} \quad \mathbb{P}\left[\cdot \,|\, \mathcal{F}_\infty^X\right] \tag{3.2.3}$$

and

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{N_t^{(m)}}{m} \leq x\right] = \mathbb{P}\left[p(Z_t) \leq x\right] . \tag{3.2.4}$$

Proof. Given the construction in (3.1.1) it is clear that conditional on the information $\mathcal{F}_\infty^X$ the defaults times $\tau_1, \tau_2 \dots, \tau_m$ are conditional independent, given $\mathcal{F}_\infty^X$. Consequently, for each $t > 0$ the Bernoulli random varialbes $1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}}, \dots, 1_{\{\tau_m \leq t\}}$ will be conditional independent given $\mathcal{F}_\infty^X$. Since the credit portfolio is homogeneous credit portfolios we have $\lambda_i(X_t) = \lambda(X_t)$ and due to the conditional independence it is also clear that conditional on $\mathcal{F}_\infty^X$ then $1_{\{\tau_1 \leq t\}}, \dots, 1_{\{\tau_m \leq t\}}$ will be i.i.d Bernoulli random variables all with the same conditional probability, that is

$$\mathbb{P}\left[1_{\{\tau_i \leq t\}} = 1 \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[\tau_i \leq t \,|\, \mathcal{F}_\infty^X\right] = 1 - \exp\left(-\int_0^t \lambda(X_u)du\right) \tag{3.2.5}$$

where the last equality follows from (3.1.2) where we let $T \to \infty$. For notational convenience we define the non-negative random variables Z_t and $p(Z_t)$ as

$$Z_t = \int_0^t \lambda(X_u)du \quad \text{and} \quad p(Z_t) = 1 - e^{-Z_t} . \tag{3.2.6}$$

So in view of (3.2.5)-(3.2.6) we thus have

$$p(Z_t) = 1 - e^{-Z_t} = 1 - e^{-\int_0^t \lambda(X_u)du} = \mathbb{P}\left[\tau_i \leq t \,|\, \mathcal{F}_\infty^X\right] \tag{3.2.7}$$

that is, $p(Z_t)$ is the conditional (i.e. random) default probability at time t for each obligor i with default time τ_i, given the information $\mathcal{F}_\infty^X$. Next, we note that for a fixed time point t, we have a so called binomial mixing model with the Bernoulli variables $1_{\{\tau_i \leq t\}}$, see e.g in Frey & McNeil (2002), Frey & McNeil (2003) and Chapter 8.4 in McNeil et al. (2005). More specific, conditionally on $\mathcal{F}_\infty^X$, and for a fixed t, we have that the law of large numbers hold, since condition on $\mathcal{F}_\infty^X$, then $1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}}, \dots, 1_{\{\tau_m \leq t\}}$ are i.i.d with default probability $p(Z_t)$, that is

$$\lim_{m\to\infty} \frac{N_t^{(m)}}{m} = p(Z_t) \quad \text{a.s. under the random measure} \quad \mathbb{P}\left[\cdot \,|\, \mathcal{F}_\infty^X\right] \tag{3.2.8}$$

see e.g. Proposition 4.5 in Frey & McNeil (2003), Proposition 8.15 in McNeil et al. (2005), p.218 in Lando (2004), or Proposition 9.10 in Hult, Lindskog, Hammarlid & Rehn (2012), which proves (3.2.8). Finally, (3.2.8) immediately implies that

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{N_t^{(m)}}{m} \leq x\right] = \mathbb{P}\left[p(Z_t) \leq x\right]$$

which proves (3.2.4) and this concludes the proof.

□

Remark 3.2. Note that in Equation (3.2.7) we can for any $u \geq t$ replace $\mathcal{F}_\infty^X$ with the information $\mathcal{F}_u^X$ since

$$\mathbb{E}\left[p(Z_t) \,|\, \mathcal{F}_u^X\right] = \mathbb{P}\left[\mathbb{P}\left[\tau_i \leq t \,|\, \mathcal{F}_\infty^X\right] \,\middle|\, \mathcal{F}_u^X\right] = \mathbb{P}\left[\tau_i \leq t \,|\, \mathcal{F}_u^X\right] = p(Z_t)$$

where the last equality is due to (3.1.2) and the fact that $u \geq t$, the second last equality is due to the tower property and that $u \geq t$.

There are two main consequences from Proposition 3.1, which we will discuss in the below two subsections.

3.2.1. ***The first consequence of Proposition 3.1***. The first consequence of Proposition 3.1 is that, for a homogeneous credit portfolio with m entities with default intensities $\lambda_i(X_t) = \lambda(X_t)$ and default times τ_i generated via (3.1.1), then conditional on the information $\mathcal{F}_\infty^X$ the random variable $N_t^{(m)}$ will for each fixed t be a mixed binomial random variable. To see this, note that conditional on $\mathcal{F}_\infty^X$ then $\{1_{\{\tau_i \leq t\}}\}_{i=1}^m$ is an i.i.d Bernoulli sequence with probability $p(Z_t)$, that is, each Bernoulli variable $1_{\{\tau_i \leq t\}}$ have, conditional on $\mathcal{F}_\infty^X$, a success probability of $p(Z_t)$ where $p(Z_t)$ is defined as in (3.2.1). Since $N_t^{(m)}$ is a sum of the variables $\{1_{\{\tau_i \leq t\}}\}_{i=1}^m$ it will, by definition, therefore conditional on $\mathcal{F}_\infty^X$, for a fixed t, then be a mixed binomial random variable with parameters m and $p(Z_t)$, so that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \binom{m}{k} p(Z_t)^k \left(1 - p(Z_t)\right)^{m-k} \tag{3.2.9}$$

and thus

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \mathbb{E}\left[\binom{m}{k} p(Z_t)^k \left(1 - p(Z_t)\right)^{m-k}\right] \tag{3.2.10}$$

where $p(Z_t)$ is defined as in (3.2.1). For more about so called mixed binomial random variables, see e.g. in Frey & McNeil (2002), Frey & McNeil (2003), Chapter 8.4 in McNeil et al. (2005) and Chapter 9.2 in Lando (2004). Hence, from (3.2.9)-(3.2.10) we see that

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \mathbb{E}\left[\binom{m}{k} p(Z_t)^k \left(1 - p(Z_t)\right)^{m-k}\right] = \int_0^\infty \binom{m}{k} p(z)^k \left(1 - p(z)\right)^{m-k} f_{Z_t}(z)\, dz \tag{3.2.11}$$

where $p(Z_t)$ is defined as in (3.2.1) and $f_{Z_t}(z)$ is the density to the random variable Z_t defined as in (3.2.1). Hence, finding $\mathbb{P}\left[N_t^{(m)} = k\right]$ via (3.2.11) will in turn again require the density $f_{Z_t}(z)$ to the random variable Z_t for each fixed t which can efficiently be obtained with e.g. Fourier inversion techniques or the saddelpoint approach, due to the fact that we often have highly analytical expression for the moment generating function to the random variable $Z_t = \int_0^t \lambda(X_s)\, ds$, in terms of the parameters describing the individual default intensity $\lambda(X_t)$. However, we here remark that a direct computation of the integral in the right hand side of (3.2.11) will for say, $m \geq 55$ in practice not work since the binomial coefficient will be to large to be stored with exact accuarcy on a standard computer, using standard math software such as e.g. `Matlab`, `R` or Python. The numerical problem arising from (3.2.11) has also been discussed in other papers, for example on p.357 in Papageorgiou & Sircar (2009). More specific, on p.357 in Papageorgiou & Sircar (2009) the authors expand the left hand side of (3.2.11), that is, the term $\mathbb{E}\left[\binom{m}{k} p(Z_t)^k \left(1 - p(Z_t)\right)^{m-k}\right]$, to obtain the formula

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \mathbb{E}\left[p(Z_t)^{m-j}\right] \tag{3.2.12}$$

and Papageorgiou & Sircar (2009) then remark that the terms in the sum (called Euler-Maclaurin sum) in the right hand side of (3.2.12) have altering signs which will lead to numerical instabilities as m increases. Furthermore, Papageorgiou & Sircar (2009) also claims that *"Even though there exist efficient algorithms for accurate numerical computation of these summations, it is necessary to limit the number of firms in the portfolio, m, to 30."*

Numerically, one can in most software packages circumvent the problem discussed in connection with Equation (3.2.11) by e.g. use built-in functions for the probability distribution of a binomial distribution applied to the "fixed" probability $p(z)$ and then repeat this for every z in the numerical quadrature of the integral in the right hand side of (3.2.11). However, as will be seen in Section 11, using built-in numerical functions such as e.g. `binopdf` in matlab, with numerical quadrature of (3.2.11) is not always reliable and will often fail for different k-values. Another drawback of this alternative and purely numerical approach is that it lacks analytical properties for the tail probability $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in the sense that we don't have an semi-analytical formula for $\mathbb{P}\left[N_t^{(m)} \geq k\right]$, which will be the case with the saddlepoint approach, up to an error of the order $O(m^{-3/2})$.

In Section 4 we use the saddlepoint approach to derive very sharp approximations to the conditional tail-probabilities for the number of defaults distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in an intensity based credit portfolio model as outlined in Section 3. Section 5 discuss how the results in Section 4 also will hold for factor copula models, such as the Gaussian one-factor model and the the Clayton copula.

3.2.2. ***The second consequence of Proposition 3.1***. The second important implication from Proposition 3.1 is that (3.2.8) in Proposition 3.1 implies that when m is large, then

$$N_t^{(m)} \overset{d}{\approx} |_{\mathcal{F}_\infty^X} \; mp(Z_t) \quad \text{for large } m \tag{3.2.13}$$

where $\overset{d}{\approx} |_{\mathcal{F}_\infty^X}$ means "approximately equal in distribution under the measure $\mathbb{P}\left[\cdot \,|\, \mathcal{F}_\infty^X\right]$". Thus, a consequence of (3.2.13) is that if m is large we have for any well defined mapping $g : \mathbb{R} \mapsto \mathbb{R}$ that

$$\mathbb{E}\left[g\left(N_t^{(m)}\right)\right] \approx \mathbb{E}\left[g\left(mp(Z_t)\right)\right] \quad \text{when } m \text{ is large.} \tag{3.2.14}$$

The approximation of $\frac{N_t^{(m)}}{m}$ via $p(Z_t)$ in (3.2.8) or $N_t^{(m)}$ via $mp(Z_t)$ in (3.2.13) and e.g. (3.2.14) is often denoted by the *large portfolio approximations (LPA)*, or sometimes just LPA-formula. The benefit of the approximation (3.2.14) is that in some models it is sometimes easier to compute the quantity $\mathbb{E}\left[g\left(mp(Z_t)\right)\right]$ compared with computing the exact quantity $\mathbb{E}\left[g\left(N_t^{(m)}\right)\right]$ is in general a tedious problem numerically as discussed in Subsection 3.2.1 above. The drawback with the LPA-method (3.2.14) is that it often requires quite high levels of m to obtain a good accuracy.

In this paper we will not focus on the LPA-approximations (3.2.13)-(3.2.14) but instead focus on the exact distribution of $N_t^{(m)}$ for arbitrary values of m (i.e. both small or large m-values), considered in Subsection 3.2.1 via the saddlepoint approach, which will be discussed in the next section, that is Section 4.

4. The saddlepoint approach for homogeneous credit portfolio distributions in conditional independent intensity based models

In this section we will use the saddlepoint approach to find very sharp approximations to the conditional probabilities $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right]$ and $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$ where $N_t^{(m)}$ is defined as in (2.1), that is $N_t^{(m)}$ is a sum of the indicator functions $1_{\{\tau_1 \leq t\}}, \ldots, 1_{\{\tau_m \leq t\}}$ where the default times $\tau_1, \tau_2 \ldots, \tau_m$ are generated from conditional independent intensity based models with identical intensities $\lambda_t = \lambda(X_t)$, as specified in Section 3 and $\mathcal{F}_t^X$ is the filtration generated by $\lambda_t = \lambda(X_t)$. We first in Subsection 4.1 briefly recap the most important features of the saddlepoint approach, which also introduces notation that will be used in the rest of this paper. Then in Subsection 4.2 we show how the saddlepoint approach can efficiently be applied to the standard binomial model. Next, in Subsection 4.3 we then use a conditional version of the results in Subsection 4.2 and apply it to credit portfolios, which is one of the main results in this paper.

4.1. **A brief recapitulation of the saddlepoint approach for lattice variables.** In this subsection we give a very short recapitulation of the most important features of the saddlepoint approach needed for this paper, and it will also introduces notation that will be used in the rest of the paper. We will mainly focus on approximations of tail-distributions to discrete random variables (i.e. for lattice variables), and leaving out the discussion of the corresponding approximation formulas for continuous random variables, since this is not needed in our setting. We stress that similar expressions for the tail distribution presented here also exists for continuous random variables.

Let X be a random variable, and let $M_X(s)$ and $K_X(s)$ be its moment-generation function and cumulative generating functions, that is

$$M_X(s) = \mathbb{E}\left[e^{sX}\right] \tag{4.1.1}$$

and

$$K_X(s) = \ln M_X(s) \tag{4.1.2}$$

where we assume that $M_X(s)$ and $K_X(s)$ are well defined for some suitable range of s, which is standard to do in the saddlepoint literature, see e.g the comments after Equation (2.1) on p.38 in Daniels (1987). Let $K_X^{'}(s), K_X^{''}(s)$ and $K_X^{'''}(s)$ be the first, second and third derivatives of $K_X(s)$ with respect to s. Next, let $\hat{s} = \hat{s}(x)$ be the solution to the saddlepoint equation $K_X^{'}(s) = x$, that is

$$K_X^{'}\left(\hat{s}\right) = x\,. \tag{4.1.3}$$

where we for notational convenience often write $\hat{s}$ for $\hat{s}(x)$ so that e.g $K_X^{''}\left(\hat{s}(x)\right)$ and $K_X^{'}\left(\hat{s}(x)\right)$ is rewritten as $K_X^{''}\left(\hat{s}\right)$ and $K_X^{'}\left(\hat{s}\right)$, and similar for the higher order derivatives of $K_X(s)$ evaluated at the point $\hat{s} = \hat{s}(x)$. Furthermore, let $\mathrm{sgn}(x)$ be the sign of x so that $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$ and $\mathrm{sgn}(x) = 0$ if $x = 0$ and let $\varphi(x)$ and $\Phi(x)$ denote the density and distribution function to a standard

normal random variable. Let $X_1, \ldots, X_m$ be an i.i.d sample with same distribution as X above and let $\bar{X}_m$ be the sample mean, that is

$$\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i \,. \tag{4.1.4}$$

Then the saddlepoint approach implies that the tail probability for the sample mean $\bar{X}_m$, that is $\mathbb{P}\left[\bar{X}_m \geq x\right]$ can for $x \neq \mathbb{E}\left[\bar{X}_m\right] = \mathbb{E}\left[X\right]$ be approximated via the so called Lugannani-Rice formula, given by

$$\mathbb{P}\left[\bar{X}_m \geq x\right] = 1 - \Phi\left(\hat{w}\right) + \varphi\left(\hat{w}\right)\left(\frac{1}{\left(1-e^{-\hat{s}}\right)\sqrt{mK_X''\left(\hat{s}\right)}} - \frac{1}{\hat{w}} + O(m^{-3/2})\right) \tag{4.1.5}$$

where $\hat{w} = \hat{w}(x)$ is given by

$$\hat{w} = \operatorname{sgn}(\hat{s})\sqrt{2m\left(x\hat{s} - K_X\left(\hat{s}\right)\right)} \tag{4.1.6}$$

see e.g. Equation (6.5) on p.44 in Daniels (1987), or Equation (3.3.17) on p.79 in Jensen (1995), where $\hat{s} = \hat{s}(x)$ and $\hat{w} = \hat{w}(x)$ are defined as in (4.1.3) and (4.1.6). If $x = \mathbb{E}\left[\bar{X}_m\right] = \mathbb{E}\left[X\right]$ there exists various so called continuity corrections to the expression $\mathbb{P}\left[\bar{X}_m \geq \mathbb{E}\left[X\right]\right]$ in (4.1.5), similar to the one in Equation (**??**), see e.g. Subsection 1.2.3 - 1.2.7 on pp.17-28 in Butler (2007).

Often, very good approximations to $\mathbb{P}\left[\bar{X}_m \geq x\right]$ can be obtained by dropping the higher order terms. For example, a good approximation to $\mathbb{P}\left[\bar{X}_m \geq x\right]$ for $x \neq \mathbb{E}\left[\bar{X}_m\right] = \mathbb{E}\left[X\right]$ is obtained from (4.1.5) as

$$\mathbb{P}\left[\bar{X}_m \geq x\right] \approx 1 - \Phi\left(\hat{w}\right) + \varphi\left(\hat{w}\right)\left(\frac{1}{\left(1-e^{-\hat{s}}\right)\sqrt{mK_X''\left(\hat{s}\right)}} - \frac{1}{\hat{w}}\right) \tag{4.1.7}$$

where $\hat{s} = \hat{s}(x)$ and $\hat{w} = \hat{w}(x)$ are defined as above.

4.2. **The saddlepoint approach applied to the binomial distribution.** If $X_1, \ldots, X_m$ is an i.i.d sample with same distribution as a Bernoulli distributed random variable X where $\mathbb{P}\left[X = 1\right] = p$ and $\mathbb{P}\left[X = 0\right] = 1 - p$ then $m\bar{X}_m = \sum_{i=1}^{m} X_i$ will be binomial distributed random variable with parameters m and p. Furthermore, straightforward computations gives

$$M_X(s) = (1-p) + pe^s \quad \text{and} \quad K_X'\left(s\right) = \frac{pe^s}{1-p+pe^s} \quad \text{and} \quad K_X''\left(s\right) = \frac{p(1-p)e^s}{\left(1-p+pe^s\right)^2} \tag{4.2.1}$$

where we remind that $K_X(s) = \ln M_X(s)$. The solution $\hat{s} = \hat{s}(x)$ to the equation $K_X'\left(\hat{s}\right) = x$ is then given by

$$e^{\hat{s}} = \frac{x}{1-x}\frac{1-p}{p} \quad \text{or, equivalently} \quad \hat{s} = \ln\left(\frac{x(1-p)}{(1-x)p}\right) \quad \text{for } 0 < x < 1 \tag{4.2.2}$$

and some calculations gives that $K_X''\left(\hat{s}\right) = x(1-x)$. Note that $\bar{X}_m$ by construction will have support on $0 \leq x \leq 1$.

By using (4.2.1), $\hat{s}(x)$ as in (4.2.2) and that $K_X''\left(\hat{s}\right) = x(1-x)$, and letting $s = \hat{s} = \hat{s}(x)$ in (4.1.5) together with some computations, one can derive very explicit formulas for sharp approximations to the tail probability $\mathbb{P}\left[\bar{X}_m \geq x\right]$ for any $0 < x < 1$, that is, it is possible to derive a approximation for the tail probability of a binomial distribution via the Lugannani-Rice formula for lattice random variables (4.1.5), see e.g. Example 3.3.4 on p.86 in Jensen (1995),

$$\mathbb{P}\left[\bar{X}_m \geq x\right] = 1 - \Phi\left(\hat{w}_B\right) + \varphi\left(\hat{w}_B\right)\left(\frac{1}{\hat{z}_B} - \frac{1}{\hat{w}_B} + O(m^{-3/2})\right) \tag{4.2.3}$$

where $\hat{w}_B = \hat{w}_B(x,m,p)$ and $\hat{z}_B = \hat{z}_B(x,m,p)$ are defined as

$$\hat{w}_B = \hat{w}_B(x,m,p) = \operatorname{sgn}\left(\hat{w}_B\right)\sqrt{2m\left(x\ln\left(\frac{x(1-p)}{(1-x)p}\right) - \ln\left(\frac{1-p}{1-x}\right)\right)} \tag{4.2.4}$$

with

$$\operatorname{sgn}\left(\hat{w}_B\right) = \operatorname{sgn}\left(\frac{x(1-p)}{(1-x)p} - 1\right) \tag{4.2.5}$$

and

$$\hat{z}_B = \hat{z}_B(x,m,p) = \sqrt{mx(1-x)}\left(1 - \frac{(1-x)p}{x(1-p)}\right) . \tag{4.2.6}$$

The equations (4.2.4)-(4.2.6) can also be found in e.g. Example 3.3.4 on p.86 in Jensen (1995). Furthermore, a slightly different version of (4.2.4)-(4.2.6) is obtained by combing Example 1.2.4(2) on p.20 in Butler (2007) together with formulas (1.27)-(1.28) on pp.17-18 in Butler (2007).

We will always assume that $x \neq p$ when using the formulas (4.2.3)-(4.2.6). If $x = p$ then $\hat{w}_B = 0$ in (4.2.4) and $\hat{z}_B = 0$ in (4.2.6) and the formula (4.2.3) can then not be used, instead we use one of the so called continuity corrections to the expression $\mathbb{P}\left[\bar{X}_m \geq \mathbb{E}[X]\right]$ in see e.g. Subsection 1.2.3 - 1.2.7 on pp.17-28 in Butler (2007). Also note that if $x = 1$ then $\hat{z}_B = 0$ and $\hat{w}_B$ is not defined, so the formula (4.2.3) can then not be used directly. However, since $\mathbb{P}\left[\bar{X}_m \geq 1\right] = p^m$, it is easy to overcome this problem and this will be discussed in more detail in Subsection 4.3.

For $0 < x < 1$ with $x \neq p$, define the function $H_B^{(\mathrm{LR})}(x, m, p)$ as

$$H_B^{(\mathrm{LR})}(x, m, p) = 1 - \Phi(\hat{w}_B) + \varphi(\hat{w}_B)\left(\frac{1}{\hat{z}_B} - \frac{1}{\hat{w}_B}\right) \tag{4.2.7}$$

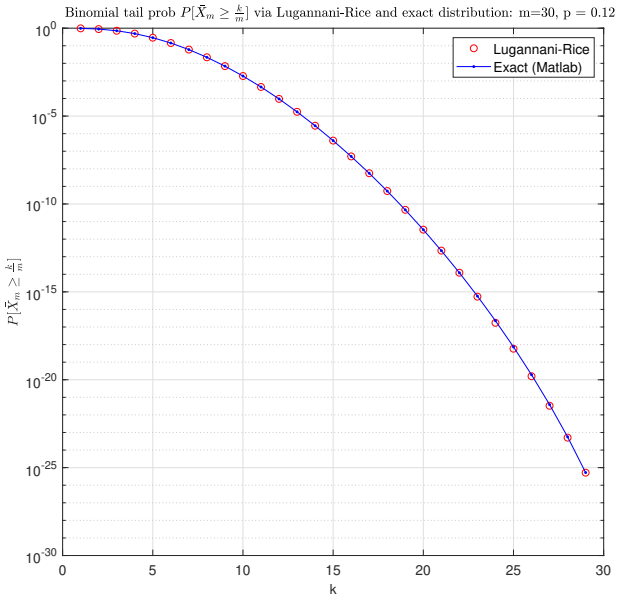
where $\hat{s} = \hat{s}(x, p)$ is defined as in Equation (4.2.2) and $\hat{w}_B = \hat{w}_B(x, p, m)$ and $\hat{z}_B = \hat{z}_B(x, p, m)$ are specified as in (4.2.4)-(4.2.6). Then, (4.2.3) and (4.2.7) implies that

$$\mathbb{P}\left[\bar{X}_m \geq x\right] \approx H_B^{(\mathrm{LR})}(x, m, p)\ . \tag{4.2.8}$$

The left panels of Figure 1 and 2 display the tail-probability $\mathbb{P}\left[\bar{X}_m \geq x\right]$ for $x = \frac{k}{m}$ and $k = 1, 2, \ldots, m-1$ for $m = 30, p = 0.12$ (in Figure 1) and $m = 125, p = 0.0329$ (in Figure 2) computed via the approximation $H_B^{(\mathrm{LR})}(x, m, p)$ in (4.2.8) calculated via (4.2.7), that is, we use (4.2.3)-(4.2.6) without the error term in (4.2.3). In the same left panels of Figure 1 and 2 we also display the $\mathbb{P}\left[\bar{X}_m \geq x\right]$ computed via the `Matlab`-function `binopdf`. More specific, by using that $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right] = \sum_{j=k}^{m} \mathbb{P}\left[\bar{X}_m = \frac{j}{m}\right] = \sum_{j=k}^{m} \mathbb{P}[Y = j]$ where $Y \stackrel{d}{=} \mathrm{Bin}(m, p)$ we can compute $\mathbb{P}\left[\bar{X}_m \geq x\right]$ for $x = \frac{k}{m}$ via the matlab-function `binopdf` which gives the probability function for a binomially distributed random variable $Y \stackrel{d}{=} \mathrm{Bin}(m, p)$. The right panels in Figure 1 and 2 shows the relative error in percent between the two methods where the relative difference is measured with respect to the `Matlab`-method.

First, in Figure 1 where $m = 30, p = 0.12$, we see that the saddlepoint method will give an extremely good fit for $1 \leq k \leq 22$ where the relative error (in percent) compared with the Matlab-method, never exceeds 0.81% which thus are very sharp approximations to the exact method which uses a sum of the matlab `binopdf`-function. At $k = 23$ the error jumps to 6.74% and for $24 \leq k \leq 28$ the error will be decreasing from 23.5% at $k = 24$ down to 9.5% at $k = 28$, and at $k = 29$ the error is only 1.45%. Furthermore, the values for $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right]$ computed via `Matlab` for $24 \leq k \leq 28$ will be decreasing from 2.27×10^{-17} down to 5.61×10^{-24} which are extremely small numbers. Thus, the saddlepoint for $24 \leq k \leq 28$ will give errors/deviations between 23.5% to 9.5% on probabilities in the range 2.27×10^{-17} down to 5.61×10^{-24} and in many practical financial/economical situations these errors will therefore not have a huge impact.

Next, in Figure 2 we repeat the same study as in Figure 1 but now for $m = 125, p = 0.0329$ (the values $m = 125, p = 0.0329$ will also be used in our numerical studies in Section 8-11, for intensity based models and factor copula settings). So regarding the accuracy for $m = 125, p = 0.0329$, we see in the right panel of Figure 2 that for $1 \leq k \leq 27$ the relative error (in percent) compared with the Matlab-method, never exceeds 0.86% which thus are very sharp approximations to the exact Matlab-method. At $k = 28$ the error jumps to 1.25% and for $29 \leq k \leq 115$ the error will be continuously decreasing from 50.0% at $k = 29$ down to 10.6% at $k = 115$. Furthermore, for $116 \leq k \leq 124$ the relative error will vary in the range of 0.89% up to 9.85%. Finally, as seen in the left panel in Figure 2, the values for $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right]$ computed via `Matlab` for $29 \leq k \leq 124$ will be continuously decreasing from 9.67×10^{-17} down to 1.64×10^{-182} which again are extremely small numbers, and as mentioned above, in most situations, errors of around 10-45% on such small numbers will in most financial/economical practical applications not have a huge impact.



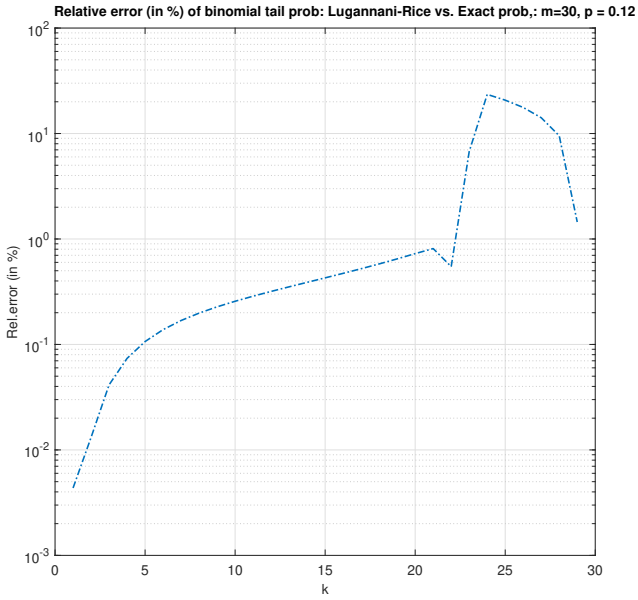
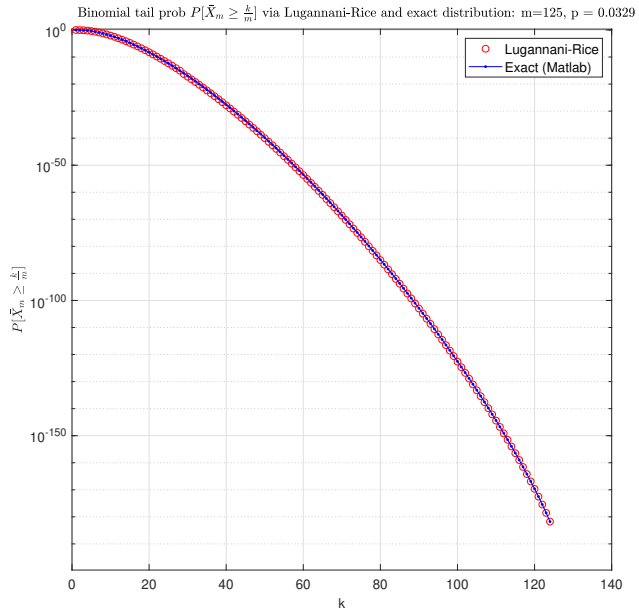


Figure 1. $m = 30, p = 0.12$. **Left panel:** Binomial tail-probability $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right]$ for $k = 1, 2, \ldots, m-1$ via Lugannani-Rice and exact (Matlab) method. **Right panel:** relative error in % where the error is measured with respect to the Matlab-method.



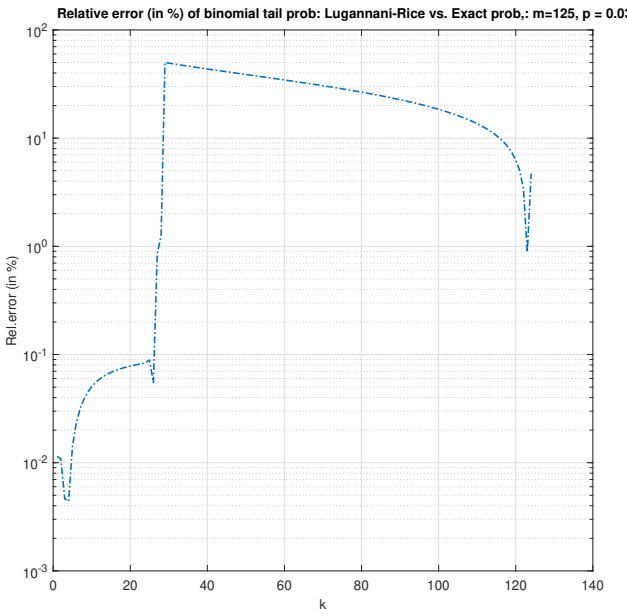


Figure 2. $m = 125, p = 0.0329$. **Left panel:** Binomial tail-probability $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right]$ for $k = 1, 2, \ldots, m-1$ via Lugannani-Rice and exact (Matlab) method. **Right panel:** relative error in % where the error is measured with respect to the Matlab-method.

Note that in Figure 1 and 2 we compute $H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p\right)$ for $k = 1, \ldots, m-1$ in (4.2.8) for the same *fixed* probability $p \in (0,1)$. In Section 4.3 and Section 5-11, we will for $k = 1, \ldots, m-1$ compute $\mathbb{E}\left[H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right)\right]$ where $p(Z_t) \in [0,1]$ and Z_t is some random variable. Hence, when computing $\mathbb{E}\left[H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right)\right]$ we will have a weighted average of $H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right)$ over different (conditional) probabilities $p(Z_t)$ as opposed to a fixed p used in Figure 1 and 2. This averaging over different values of $p(Z_t)$ in $\mathbb{E}\left[H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right)\right]$ will significantly reduce the corresponding relative errors compared with

the "exact" method, when computing the tail-probability $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ of the number of defaults in a intensity-based credit risk portfolio model or a factor copula model, as will be seen in Section 11 (see for example in Figure 20 and Figure 21 in Section 11). Hence, the corresponding errors shown in Figure 1 and 2 will in a conditional independent credit risk model (using mixed binomial models), decrease significantly in particular for moderate or large k-values, and in fact, the Matlab-method (via the `binopdf`-function) will very often fail numerically while the saddlepoint method is very robust, see e.g. in Figure 22 and Figure 23 in Section 11.

Before we end this subsection, we give a final comment regarding the exact method in Figure 1 and 2. Note that another alternative to the sum of `binopdf`-functions when computing $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right]$ in Figure 1 and 2 is to use the distribution function `binocdf` to $Y \stackrel{d}{=} \text{Bin}(m,p)$ in Matlab since $\mathbb{P}\left[\bar{X}_m \geq \frac{k}{m}\right] = 1 - \mathbb{P}\left[Y \leq k-1\right]$. But using this second method via the distribution function `binocdf` is a much worse option than `binopdf`, since `Matlab` will already for moderate sizes of k treat the value $1-\mathbb{P}\left[Y \leq k-1\right]$ as zero due to floating point relative accuracy, that is, for moderate k-values $\mathbb{P}\left[Y \leq k-1\right]$ will be extremely close to one so that Matlab then set $1 - \mathbb{P}\left[Y \leq k-1\right]$ to zero (the same numerical phenomena will also hold in R, Python and Excel).

4.3. **The saddlepoint approach applied to the conditional binomial distribution.** We will in this subsection use a conditional version of the results in Subsection 4.2 and apply it to credit portfolios in an intensity based setting as presented in Section 3.

We will below use the similar notation for sample means as in Subsection 4.2, and therefore we will from now on, unless explicitly stated often let $\frac{N_t^{(m)}}{m}$ be denoted by $\bar{N}_t^{(m)}$, that is

$$\bar{N}_t^{(m)} = \frac{1}{m} N_t^{(m)}. \tag{4.3.1}$$

and with the notation in (4.3.1) we then for any integer k with $k \leq m$ have that

$$\mathbb{P}\left[N_t^{(m)} \geq k\right] = \mathbb{P}\left[\bar{N}_t^{(m)} \geq k/m\right]. \tag{4.3.2}$$

With the notation and setup from Subsection 4.2, we next state the following theorem, which gives analytical expressions for the tail probabilities of $\frac{N_t^{(m)}}{m}$ where $N_t^{(m)}$ is defined as in Subsection 3.2.

Theorem 4.1. *Consider a homogeneous credit portfolio with m entities with default intensities $\lambda_i(X_t) = \lambda(X_t)$ and default times τ_i generated via* (3.1.1)*. Define Z_t and $p(Z_t)$ as in* (3.2.1)*. Then, with notation as above, for $0 \leq x \leq 1$*

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right] = H_B^{(LR)}\left(x, m, p(Z_t)\right) + O(m^{-3/2}). \tag{4.3.3}$$

and thus

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x\right] = \mathbb{E}\left[H_B^{(LR)}\left(x, m, p(Z_t)\right)\right] + O(m^{-3/2}) \tag{4.3.4}$$

where

$$\mathbb{E}\left[H_B^{(LR)}\left(x, m, p(Z_t)\right)\right] = \int_0^\infty H_B^{(LR)}\left(x, m, p(z)\right) f_{Z_t}(z)\, dz \tag{4.3.5}$$

and $f_{Z_t}(z)$ is the density to the random variable Z_t defined as in (3.2.1)*. Furthermore, the mapping $H_B^{(LR)}(x,m,p)$ is for $0 < x < 1$ defined as*

$$H_B^{(LR)}(x,m,p) = 1 - \Phi\left(\hat{w}_B\right) + \varphi\left(\hat{w}_B\right)\left(\frac{1}{\hat{z}_B} - \frac{1}{\hat{w}_B}\right) \tag{4.3.6}$$

where $\Phi(x)$ and $\varphi(x)$ are the distribution function and density to a standard normal random variable and for $0 < x < 1$ and $x \neq p$ we have

$$\hat{w}_B = \hat{w}_B(x,m,p) = sgn\left(\hat{w}_B\right)\sqrt{2m\left(x \ln\left(\frac{x(1-p)}{(1-x)p}\right) - \ln\left(\frac{1-p}{1-x}\right)\right)} \tag{4.3.7}$$

with

$$sgn\left(\hat{w}_B\right) = sgn\left(\frac{x(1-p)}{(1-x)p} - 1\right) \tag{4.3.8}$$

where

$$\hat{z}_B = \hat{z}_B(x,m,p) = \sqrt{mx(1-x)}\left(1 - \frac{(1-x)p}{x(1-p)}\right) \tag{4.3.9}$$

for $0 < x < 1$ *and* $x \neq p$. *Finally, for* $x = 0$ *and* $x = 1$ *we define the mapping* $H_B^{(LR)}(x, m, p)$ *as*

$$H_B^{(LR)}(0, m, p) = 1 \quad \textit{and} \quad H_B^{(LR)}(1, m, p) = p^m \,. \tag{4.3.10}$$

Proof. As pointed out in Equation (3.2.10), for a homogeneous credit portfolio with m entities with default intensities $\lambda_i(X_t) = \lambda(X_t)$ and default times τ_i generated via (3.1.1), then conditional on the information $\mathcal{F}_\infty^X$ the random variable $N_t^{(m)}$ will for each fixed t be a mixed binomial random variable. Hence, conditional on the information $\mathcal{F}_\infty^X$, and for a fixed t, we can repeat the saddlepoint procedure for binomial models presented in Subsection 4.2. More specific, in Subsection 4.2 let X_i be replaced by $1_{\{\tau_i \leq t\}}$, $\bar{X}_m$ with $\frac{N_t^{(m)}}{m}$ and p with $p(Z_t)$, where τ_i is generated via (3.1.1) for default intensities $\lambda_i(X_t) = \lambda(X_t)$ and $p(Z_t)$ is defined as in (3.2.1) and $N_t^{(m)}$ is given by (2.1), that is $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Next, let τ have the same distribution as $\tau_1, \ldots, \tau_m$ in our homogeneous credit portfolio. Then, for a fixed t, we can define a conditional moment-generation function and cumulative generating functions $M_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right)$ and $K_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right)$ to the Bernoulli random variable $1_{\{\tau \leq t\}}$ as

$$M_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = \mathbb{E}\left[e^{s 1_{\{\tau \leq t\}}} \,\middle|\, \mathcal{F}_\infty^X \right] \tag{4.3.11}$$

and

$$K_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = \ln M_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) \tag{4.3.12}$$

where we assume that $M_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right)$ and $K_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right)$ are well defined for some suitable range of s, which is standard to do in the saddlepoint literature, see e.g the comments after Equation (2.1) on p.38 in Daniels (1987). Just as (4.2.1)-(4.2.2), straightforward computations gives

$$M_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = 1 - p(Z_t) + p(Z_t)e^s \quad \text{and} \quad K'_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = \frac{p(Z_t)e^s}{1 - p(Z_t) + p(Z_t)e^s} \tag{4.3.13}$$

and

$$K''_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = \frac{p(Z_t)\left(1 - p(Z_t)\right)e^s}{\left(1 - p(Z_t) + p(Z_t)e^s\right)^2} \tag{4.3.14}$$

where $p(Z_t)$ is defined as in (3.2.1). Furthermore, the solution to $\hat{s} = \hat{s}(x)$ to the equation $K'_{\tau,t}\left(s \left| \mathcal{F}_\infty^X \right.\right) = x$ is then given by

$$e^{\hat{s}} = \frac{x\left(1 - p(Z_t)\right)}{(1 - x)p(Z_t)} \quad \text{or, equivalently} \quad \hat{s} = \ln\left(\frac{x\left(1 - p(Z_t)\right)}{(1 - x)p(Z_t)}\right) \quad \text{for } 0 < x < 1 \tag{4.3.15}$$

and some calculations gives that $K''_{\tau,t}\left(\hat{s} \left| \mathcal{F}_\infty^X \right.\right) = x(1 - x)$ where we remind that $\frac{N_t^{(m)}}{m}$ by construction will have support on $0 \leq x \leq 1$. Now, with the equations (4.3.13) - (4.3.15) we can, for a fixed t and conditional on the information $\mathcal{F}_\infty^X$, immediately apply a conditional version of the Lugannani-Rice formula for lattice random variables applied to the conditional binomial distribution $N_t^{(m)}$ via the Equations (4.2.3) - (4.2.6) and with the notation in (4.3.1) to obtain for $0 < x < 1$

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X \right] = H_B^{(\mathrm{LR})}\left(x, m, p(Z_t)\right) + O(m^{-3/2}) \tag{4.3.16}$$

where the function $H_B^{(\mathrm{LR})}(x, m, p)$ is defined as in (4.2.7) and with $\hat{w}_B = \hat{w}_B(x, p, m)$ and $\hat{z}_B = \hat{z}_B(x, p, m)$ are specified as in (4.2.4)-(4.2.6). Thus, (4.3.16) is a conditional version of (4.2.3)-(4.2.7) with the constant p in (4.2.3)-(4.2.7) now replaced with the random variable $p(Z_t)$ defined as in (3.2.1) and where the conditioning is done with respect to the filtration $\mathcal{F}_\infty^X$. If $x = 0$ we obviously have that $\mathbb{P}\left[\bar{N}_t^{(m)} \geq 0 \,\middle|\, \mathcal{F}_\infty^X \right] = 1$ which in view of (4.3.16) motivates why we define $H_B^{(\mathrm{LR})}(x, m, p)$ at $x = 0$ as

$$H_B^{(\mathrm{LR})}(0, m, p) = 1 \,. \tag{4.3.17}$$

Next, let us consider the case $x = 1$. First, note that

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq 1 \,\middle|\, \mathcal{F}_\infty^X \right] = \mathbb{P}\left[N_t^{(m)} = m \,\middle|\, \mathcal{F}_\infty^X \right] \tag{4.3.18}$$

and recall from (3.2.9) that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X \right] = \binom{m}{k} p(Z_t)^k \left(1 - p(Z_t)\right)^{m-k} \tag{4.3.19}$$

for every $k = 0, 1, \ldots, m$. So letting $k = m$ in (4.3.19) we get that

$$\mathbb{P}\left[N_t^{(m)} = m \,\middle|\, \mathcal{F}_\infty^X \right] = p(Z_t)^m \,. \tag{4.3.20}$$

Then, (4.3.18) and (4.3.20) imply that

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq 1 \,\middle|\, \mathcal{F}_\infty^X \right] = p(Z_t)^m \,. \tag{4.3.21}$$

which in view of (4.3.16) motivates why we define $H_B^{(\mathrm{LR})}(x, m, p)$ at $x = 1$ as

$$H_B^{(\mathrm{LR})}(1, m, p) = p^m \,. \tag{4.3.22}$$

Hence, (4.3.17) and (4.3.22) proves (4.3.10). Finally, from (4.3.16) we immediately get that

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \right] = \mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t)) \right] + O(m^{-3/2}) \tag{4.3.23}$$

where

$$\mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t)) \right] = \int_0^\infty H_B^{(\mathrm{LR})}(x, m, p(z))\, f_{Z_t}(z)\, dz \tag{4.3.24}$$

where $f_{Z_t}(z)$ is the density to the random variable Z_t defined as in (3.2.1) and $H_B^{(\mathrm{LR})}(x, m, p)$ for $0 < x < 1$ is given in (4.2.7) with $\hat{w}_B = \hat{w}_B(x, p, m)$ and $\hat{z}_B = \hat{z}_B(x, p, m)$ specified as in (4.2.4)-(4.2.6) and for $x = 0$, $x = 1$ the mapping $H_B^{(\mathrm{LR})}(x, m, p)$ is defined as in (4.3.10). This proves (4.3.3)-(4.3.10) which concludes the theorem.

□

In connection we the results in Theorem 4.1 we now make some remarks.

Remark 4.2. Note that in Theorem 4.1 and its proof we can everywhere replace $\mathcal{F}_\infty^X$ with $\mathcal{F}_t^X$ due to the same observations as in Remark 3.2. In particular, Equation (4.3.3) can then be rephrased as

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_t^X \right] = H_B^{(\mathrm{LR})}(x, m, p(Z_t)) + O(m^{-3/2}) \,. \tag{4.3.25}$$

since

$$\begin{aligned}
\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_t^X \right] &= \mathbb{E}\left[1_{\left\{ \bar{N}_t^{(m)} \geq x \right\}} \,\middle|\, \mathcal{F}_t^X \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_{\left\{ \bar{N}_t^{(m)} \geq x \right\}} \,\middle|\, \mathcal{F}_\infty^X \right] \,\middle|\, \mathcal{F}_t^X \right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X \right] \,\middle|\, \mathcal{F}_t^X \right] \\
&= \mathbb{P}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t)) \,\middle|\, \mathcal{F}_t^X \right] + O(m^{-3/2}) \\
&= H_B^{(\mathrm{LR})}(x, m, p(Z_t)) + O(m^{-3/2})
\end{aligned}$$

where the forth equality is due to (4.3.3) and the last equality follows from the fact that $H_B^{(\mathrm{LR})}(x, m, p(Z_t))$ is $\mathcal{F}_t^X$-measurable.

Remark 4.3. Note that saddlepoint approximation formulas for $\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X \right]$ and $\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \right]$ via the Equations (4.3.3) and (4.3.4)-(4.3.5) in Theorem 4.1 are specified for arbitrary values for x where $0 \leq x \leq 1$, and these approximations are continuous function in x on the interval $0 \leq x \leq 1$. However, we remind that the random variable $\bar{N}_t^{(m)}$ specified in (4.3.1) and used in Theorem 4.1 is a discrete distribution having support on the lattice $\left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, \frac{m}{m}\right\}$. The formulas (4.3.3) and and (4.3.4)-(4.3.5) in Theorem 4.1 will be decreasing and well-defined functions when x lies in the lattice $\left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, \frac{m}{m}\right\}$, which will be seen in our numerical studies in Section 8 and Section 9, but this is also clearly indicated in the standard binomial case as displayed in Figure 1 and Figure 2.

Remark 4.4. **Regarding the case $x = p$.** Observe that in Theorem 4.1 we made the assumption that $x \neq p$ when using $H_B^{(\mathrm{LR})}(x, m, p)$ given by (4.2.7) with $\hat{w}_B = \hat{w}_B(x, p, m)$ and $\hat{z}_B = \hat{z}_B(x, p, m)$ specified as in (4.2.4)-(4.2.6). In practice the case $x = p$ will never be a problem for conditional binomial models. To see this, we note that the expected value $\mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t)) \right]$ given by (4.3.24) is in practice always computed by numerical quadrature. More specific, if $f_{Z_t}(z)$ is the density to the random variable Z_t

defined as in (3.2.1), then we consider a mesh $z_1 < z_2 < \dots z_{J_C}$ and approximate $\mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t))\right]$ via the relation

$$\mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t))\right] = \int_0^\infty H_B^{(\mathrm{LR})}(x, m, p(z))\, f_{Z_t}(z)\, dz \approx \sum_{n=1}^{J_C} H_B^{(\mathrm{LR})}(x, m, p(z_n))\, f_{Z_t}(z_n)\Delta z_n. \quad (4.3.26)$$

If it happens that $x = p(z_n)$ for one of the points z_n in the mesh $z_1 < z_2 < \dots z_{J_C}$ used in (4.3.26), we can simply remove this point z_n since if the mesh is large, then removing one of the terms in the sum in (4.3.26) will hardly change the accuracy of the approximation to $\mathbb{E}\left[H_B^{(\mathrm{LR})}(x, m, p(Z_t))\right]$. Alternatively, one can replace the point z_n in the mesh it with another approximating point $\tilde{z}_n$ close to z_n such that $x \neq p(\tilde{z}_n)$. Finally, if one really want to use the mesh without changing the points in the case $x = p(z_n)$, there are ways to deal with this issue by using one of the so called continuity corrections to $H_B^{(\mathrm{LR})}(x, m, p)$, see e.g. Subsection 1.2.3 - 1.2.7 on pp.17-28 in Butler (2007).

Theorem 4.1 next implies the following corollary which allows us to efficiently compute $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$ for arbitrary $k = 0, 1, 2, \dots, m$ and this corollary also introduces notation that will be used in the rest of this paper.

Corollary 4.5. *Consider a homogeneous credit portfolio with m entities with default intensities $\lambda_i(X_t) = \lambda(X_t)$ and default times τ_i generated via (3.1.1). Define Z_t and $p(Z_t)$ as in (3.2.1). Then, with notation as above, for any integer $k = 0, 1, 2, \dots, m$*

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \Delta H_B^{(LR)}(k, m, p(Z_t)) + O(m^{-3/2}) \quad (4.3.27)$$

and thus

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \mathbb{E}\left[\Delta H_B^{(LR)}(k, m, p(Z_t))\right] + O(m^{-3/2}) \quad (4.3.28)$$

where $\Delta H_B^{(LR)}(k, m, p(Z_t))$ is defined as

$$\Delta H_B^{(LR)}(k, m, p(Z_t)) = \begin{cases} H_B^{(LR)}\left(\frac{k}{m}, m, p(Z_t)\right) - H_B^{(LR)}\left(\frac{k+1}{m}, m, p(Z_t)\right) & \text{for } k \leq m-1 \\ p(Z_t)^m & \text{for } k = m \end{cases} \quad (4.3.29)$$

and $H_B^{(LR)}(x, m, p)$ is given by (4.3.6)-(4.3.10) in Theorem 4.1.

Proof. For any integer $k = 1, 2, \dots, m-1$ we obviously have that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[N_t^{(m)} \geq k \,\middle|\, \mathcal{F}_\infty^X\right] - \mathbb{P}\left[N_t^{(m)} \geq k+1 \,\middle|\, \mathcal{F}_\infty^X\right] \quad (4.3.30)$$

and thus

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[\bar{N}_t^{(m)} \geq \frac{k}{m} \,\middle|\, \mathcal{F}_\infty^X\right] - \mathbb{P}\left[\bar{N}_t^{(m)} \geq \frac{k+1}{m} \,\middle|\, \mathcal{F}_\infty^X\right]. \quad (4.3.31)$$

Next, for $k \leq m-1$ so that $\frac{k+1}{m} \leq 1$ we can apply (4.3.3) in Theorem 4.1 to the right hand side of (4.3.31) to obtain that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right) - H_B^{(\mathrm{LR})}\left(\frac{k+1}{m}, m, p(Z_t)\right) + O(m^{-3/2}) \quad (4.3.32)$$

and $H_B^{(\mathrm{LR})}(x, m, p)$ is for $0 < x < 1$ given by Equation (4.3.6) and by (4.3.10) when $x = 0$ or $x = 1$, in Theorem 4.1. Next, for $k = m$ we obviously have that

$$\mathbb{P}\left[N_t^{(m)} = m \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[N_t^{(m)} \geq m \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[\bar{N}_t^{(m)} \geq 1 \,\middle|\, \mathcal{F}_\infty^X\right] \quad (4.3.33)$$

so $H_B^{(\mathrm{LR})}(x, m, p)$ given by (4.3.6) for $x = 1$ together with (4.3.10) in Theorem 4.1 and Equation (4.3.33) then renders that

$$\mathbb{P}\left[N_t^{(m)} = m \,\middle|\, \mathcal{F}_\infty^X\right] = H_B^{(\mathrm{LR})}(1, m, p(Z_t)) = p(Z_t)^m. \quad (4.3.34)$$

Finally, for $k = m-1$ we have, just as in (4.3.31) that

$$\mathbb{P}\left[N_t^{(m)} = m-1 \,\middle|\, \mathcal{F}_\infty^X\right] = \mathbb{P}\left[\bar{N}_t^{(m)} \geq \frac{m-1}{m} \,\middle|\, \mathcal{F}_\infty^X\right] - \mathbb{P}\left[\bar{N}_t^{(m)} \geq 1 \,\middle|\, \mathcal{F}_\infty^X\right]. \quad (4.3.35)$$

and the second term in the right hand side of (4.3.35) is computed as in (4.3.34) while the first term in the right hand side of (4.3.35) is given by $H_B^{(\mathrm{LR})}\left(\frac{m-1}{m}, m, p(Z_t)\right) + O(m^{-3/2})$ which implies that

$$\mathbb{P}\left[N_t^{(m)} = m-1 \,\middle|\, \mathcal{F}_\infty^X\right] = H_B^{(\mathrm{LR})}\left(\frac{m-1}{m}, m, p(Z_t)\right) - H_B^{(\mathrm{LR})}\left(1, m, p(Z_t)\right) + O(m^{-3/2}) \tag{4.3.36}$$

where $H_B^{(\mathrm{LR})}(x, m, p)$ is for $0 < x < 1$ given by Equation (4.3.6) in Theorem 4.1 and for $x = 1$, by (4.3.10) in Theorem 4.1, so that (4.3.36) can be restated as

$$\mathbb{P}\left[N_t^{(m)} = m-1 \,\middle|\, \mathcal{F}_\infty^X\right] = H_B^{(\mathrm{LR})}\left(\frac{m-1}{m}, m, p(Z_t)\right) - p(Z_t)^m + O(m^{-3/2})\,. \tag{4.3.37}$$

Finally, for any integers m and k with $0 < k \leq m$ and any $p \in (0,1)$ we define the mapping $\Delta H_B^{(\mathrm{LR})}(k, m, p)$ as

$$\Delta H_B^{(\mathrm{LR})}\left(k, m, p(Z_t)\right) = \begin{cases} H_B^{(\mathrm{LR})}\left(\frac{k}{m}, m, p(Z_t)\right) - H_B^{(\mathrm{LR})}\left(\frac{k+1}{m}, m, p(Z_t)\right) & \text{for } k \leq m-1 \\ p(Z_t)^m & \text{for } k = m \end{cases}. \tag{4.3.38}$$

Hence, (4.3.38) will then together with (4.3.32), (4.3.34) and (4.3.36) (or (4.3.37)) imply that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \Delta H_B^{(\mathrm{LR})}\left(k, m, p(Z_t)\right) + O(m^{-3/2})$$

which proves the formulas (4.3.27) and (4.3.29). Finally, (4.3.28) is obtained by taking expected value of (4.3.27), which concludes the corollary. □

Remark 4.6. Note that, by similar arguments as in Remark 4.2 we can everywhere replace $\mathcal{F}_\infty^X$ with $\mathcal{F}_t^X$ everywhere in Corollary 4.5, and in particular in (4.3.27) so that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_t^X\right] = \Delta H_B^{(\mathrm{LR})}\left(k, m, p(Z_t)\right) + O(m^{-3/2}) \tag{4.3.39}$$

where $\Delta H_B^{(\mathrm{LR})}(k, m, p(Z_t))$ is defined as in (4.3.29).

In Section 11 we will perform numerical testes of the algorithm in Corollary 4.5, and there we show that the saddlepoint method for $\mathbb{P}\left[N_t^{(m)} = k\right]$ is as good, of often even better than using e.g. built in software routines computing $\mathbb{P}\left[N_t^{(m)} = k\right]$.

The saddlepoint approach introduced in Theorem 4.1 and Corollary 4.5 will produce a method of order $O(1)$ when computing $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right]$ and thus also $O(1)$ for computing $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$. Hence, for a portfolio with m obligors, our approach will therefore for computing the whole distribution $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$, that is for all $k = 0, 1, \ldots, m$, be of order $O(m)$, i.e. linear in the portfolio size m, which is an dramatic improvement compared with the quadratic, $O(m^2)$, recursive algorithm introduced in Andersen et al. (2003). Another advantage with our saddlepoint approach is that it leads to explicit formulas for $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right]$ and thus also for $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$ in terms of x, k, m and the $\mathcal{F}_\infty^X$-conditional individual default probability $p(Z_t)$ defined as in (3.2.1).

Besides the recursive algorithms discussed in e.g. Andersen et al. (2003) and Andersen & Sidenius (2004), there also exists Fourier methods where the characteristic function for the loss distribution is directly derived and used with Fourier inversion methods such as the FFT method and such techniques are discussed in e.g. Gregory & Laurent (2003) and Gregory & Laurent (2005). However, as pointed out in Andersen et al. (2003), such Fourier inversion methods seems to be much slower than recursive algorithms developed in Andersen et al. (2003) and Andersen & Sidenius (2004). For example, on p.68 in Andersen et al. (2003) the authors remark that they conducted numerical examples where the Fourier method could be up to 25 times slower than the recursive algorithm. Given that our saddlepoint approach has linear complexity in terms of the portfolio size, compared to the quadratic complexity of the recursive algorithms in Andersen et al. (2003), we therefore conclude that or method should also be much faster than the Fourier inversion methods proposed in Gregory & Laurent (2003) and Gregory & Laurent (2005).

While Theorem 4.1 and Corollary 4.5 are focusing on intensity based conditional independent exchangeable models it is easy to see that the results in these theorems also will hold for other type of conditional independent exchangeable models, for example conditional factor models. This will be discussed in the next section, that is Section 5.

5. The saddlepoint approach for homogeneous credit portfolio loss distributions in conditional factor based models

In Section 4 we presented efficient formulas for the tail-distribution to the number of defaults $N_t^{(m)}$ in conditional independent homogeneous models where the default times were constructed in an intensity based settings. While Theorem 4.1 and Corollary 4.5 are focusing on intensity based conditional independent exchangeable models it is easy to see that the results in these theorems also will hold for other type of conditional independent exchangeable models, for example conditional factor models, discussed in e.g Gordy (2002), Gregory & Laurent (2005), Huang et al. (2007) and Martin (2011), which includes the dynamic Gaussian one-factor model often used in the credit literature. For such factor models, we simply replace $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, \mathcal{F}_\infty^X\right]$ and $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right]$ in Theorem 4.1 and Corollary 4.5 with $\mathbb{P}\left[N_t^{(m)} \geq x \,\middle|\, Z\right]$ and $\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, Z\right]$ for some static factor Z, and that will be the topic of this section. First, we give a brief and general discussion about factor copula models. Then, Subsection 5.1 treats the very common one-factor Gaussian copula model while Subsection 5.2 considers so called Archimedean copulas where we focus on the so-called Clayton copula. Finally, Subsection 5.3 shows how the results from Subsection 4.3 can be applied to the factor copula models presented in this section.

In this section we consider the distribution of $N_t^{(m)}$ when the default times $\tau_1, \ldots, \tau_m$ are driven by factor copulas in a homogeneous credit portfolio, meaning that there exists a time-invariant random variable Z and a mapping $p(t,z)$ such that $0 \leq p(t,z) \leq 1$ for all $t \geq 0$ and $z \in \mathbb{R}$ with

$$\mathbb{P}\left[\tau_i \leq t \,|\, Z\right] = p(t, Z) \tag{5.1}$$

for all obligors i and conditional on Z then $\tau_1, \ldots, \tau_m$ are independent, that is

$$\mathbb{P}\left[\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_m \leq t_m \,|\, Z\right] = \prod_{i=1}^{m} \mathbb{P}\left[\tau_i \leq t_i \,|\, Z\right] \tag{5.2}$$

for all $t_1 \geq 0, \ldots, t_m \geq 0$. Hence, (5.1)-(5.2) implies a homogeneous conditional independent portfolio credit model and there exists several widely used such so called factor copula models. Below we specify two such factor copula models. Since we in this paper will only focus on homogeneous portfolios then the marginal default probabilities $F_i(t) = \mathbb{P}\left[\tau_i \leq t\right]$ will be same for all obligors and we will for notational convenience write $F(t) = F_i(t)$ for the marginal default probabilities at time t.

5.1. The one-factor Gaussian copula model.

The *one-factor Gaussian copula model* is probably the most famous factor copula model and it works both for heterogenous and homogeneous credit portfolios, see e.g. in Li (2000), Gregory & Laurent (2005), Gregory & Laurent (2003), Andersen & Sidenius (2004), Crépey, Jeanblanc & Wu (2013) and McNeil et al. (2005). Here we only focus on homogeneous portfolios and let $F(t)$ be the marginal default probability at time t same for obligors with default times $\tau_1, \ldots, \tau_m$, that is $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$. Let Z be an standard normal random variable. Then, in the homogeneous one-factor Gaussian copula model the conditional marginal default probabilities $\mathbb{P}\left[\tau_i \leq t \,|\, Z\right]$ satisfy

$$\mathbb{P}\left[\tau_i \leq t \,|\, Z\right] = \Phi\left(\frac{\Phi^{-1}\left(F(t)\right) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{5.1.1}$$

where ρ is the so-called correlation-parameters which is a constant such that $\rho \in [0,1]$ and $\Phi(x)$ denotes the distribution function to a standard normal random variable. Furthermore, one can show that conditional on Z then $\tau_1, \ldots, \tau_m$ are independent, that is (5.2) will hold. Thus, from (5.1.1) and in view of the notation in (5.1) we have that

$$p(t, Z) = \Phi\left(\frac{\Phi^{-1}\left(F(t)\right) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right). \tag{5.1.2}$$

For more about the *one-factor Gaussian copula model* see e.g. in Li (2000), Gregory & Laurent (2005), Gregory & Laurent (2003), Andersen & Sidenius (2004), Crépey et al. (2013) and McNeil et al. (2005).

5.2. Archimedean copulas.

There exists many types of copula functions, and applications of these in portfolio credit risk (both static and dynamic) has been a huge topic in academia and the industry, the last 25 years. In this subsection we consider so called *Archimedean copulas*, see e.g. Schönbucher (2002), Schönbucher (2003), McNeil et al. (2005), Burtschell, Gregory & Laurent (2009) or Hofert & Scherer (2011).

Here we only focus on homogeneous portfolios and let $F(t)$ be the marginal default probability at time t same for obligors with default times $\tau_1, \ldots, \tau_m$, that is $F(t) = \mathbb{P}[\tau_i \leq t]$. Let Z be a non-negative random variable with Laplace-transform $\mathcal{L}_Z(x)$ and define the function $\psi(x)$ as the inverse of $\mathcal{L}_Z(x)$, that is $\psi(x) = \mathcal{L}_Z^{-1}(x)$. Then, in the homogeneous Archimedean copula model for a homogeneous portfolio credit risk, the conditional marginal default probabilities $\mathbb{P}[\tau_i \leq t \,|\, Z]$ satisfy

$$\mathbb{P}[\tau_i \leq t \,|\, Z] = \exp(-Z\psi(F(t))) \tag{5.2.1}$$

and one can show that conditional on Z then $\tau_1, \ldots, \tau_m$ are independent, that is (5.2) will hold. Thus, from (5.2.1) and in view of the notation in (5.1) we have that

$$p(t, Z) = \exp(-Z\psi(F(t))) \,. \tag{5.2.2}$$

The choice of the non-negative variable Z is often done so that its Laplace-transform and its inverse are given in closed form, that is, $\psi(x)$ and $\psi^{-1}(x)$ are analytic. Three examples for Z are often used in the credit literature, namely the so called Clayton copula, Gumbel copula and Frank copula, see e.g. in Schönbucher (2002), Schönbucher (2003) or McNeil et al. (2005). Let us here briefly mention the Clayton copula, which often is used in CDO pricing, see e.g. in Burtschell et al. (2009) or Hofert & Scherer (2011).

5.2.1. *The Clayton copula.* Recall that a non-negative random variable is said to be a gamma-distributed with parameters a and b if its density $f(x)$ is given by $f(x) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}$ for $x \geq 0$ where $\Gamma(a)$ is the gamma function. Let Z be a gamma-distributed random variable with parameters $a = \frac{1}{\theta}$ and $b = 1$ so that its density $f_Z(z)$ is given by

$$f_Z(z) = \frac{z^{\frac{1-\theta}{\theta}}e^{-z}}{\Gamma\left(\frac{1}{\theta}\right)} \qquad \text{for } z \geq 0\,.$$

The Laplace-transform of Z is then $(1+s)^{-\frac{1}{\theta}}$ and its inverse, denoted by $\psi(x)$ is then given by $\psi(x) = \left(x^{-\theta} - 1\right)$. Now, consider a *factor copula credit model created by the Archimedean Clayton copula with generator* $\psi(x) = \left(x^{-\theta} - 1\right)$. Then, in view of (5.2.2) we get

$$\mathbb{P}[\tau_i \leq t \,|\, Z] = \exp\left(Z\left(1 - F(t)^{-\theta}\right)\right) \tag{5.2.1.1}$$

that is, in view of the notation in (5.1) we have that

$$p(t, Z) = \exp\left(Z\left(1 - F(t)^{-\theta}\right)\right) \tag{5.2.1.2}$$

see e.g. in Burtschell et al. (2009) or Hofert & Scherer (2011).

5.3. **The saddlepoint approach for homogeneous credit portfolio loss distributions in conditional factor based models.** In view of (5.1)-(5.2) we have a homogeneous conditional independent portfolio credit model, that is conditional on Z then $N_t^{(m)}$ will for a fixed timepoint t be a mixed binomial random variable, or equivalently, conditional on Z then $\{1_{\{\tau_i \leq t\}}\}_{i=1}^m$ is an i.i.d Bernoulli sequence with probability $p(t, Z)$. Hence, conditional on Z we can apply Lugannani-Rice formula for binomial distributions to find sharp approximations to $\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, Z\right]$ in terms of $x, p(t, Z), m$ and also $\mathbb{P}\left[\bar{N}_t^{(m)} \geq x\right]$ in terms of density or probability function to Z etc. We can therefore state the following corollary to Theorem 4.1 and Corollary 4.5.

Corollary 5.1. *Let Z be a random variable and consider a homogeneous credit portfolio with m entities with default times $\tau_1, \ldots, \tau_m$ that satisfy* (5.1)-(5.2) *and having conditional default probabilities given by $p(t, Z)$. Then, with notation as above, for $0 \leq x \leq$ we have*

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x \,\middle|\, Z\right] = H_B^{(LR)}(x, m, p(t, Z)) + O(m^{-3/2}) \tag{5.3.1}$$

and

$$\mathbb{P}\left[\bar{N}_t^{(m)} \geq x\right] = \mathbb{E}\left[H_B^{(LR)}(x, m, p(t, Z))\right] + O(m^{-3/2}) \tag{5.3.2}$$

where the mapping $H_B^{(LR)}(x, m, p)$ is given by (4.3.6)-(4.3.10) *in Theorem 4.1. Furthermore, for any integer k such that $0 \leq k \leq m$ we have*

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, Z\right] = \Delta H_B^{(LR)}(k, m, p(t, Z)) + O(m^{-3/2}) \tag{5.3.3}$$

and thus

$$\mathbb{P}\left[N_t^{(m)}=k\right]=\mathbb{E}\left[\Delta H_B^{(LR)}\left(k,m,p(t,Z)\right)\right]+O(m^{-3/2}) \tag{5.3.4}$$

where $\Delta H_B^{(LR)}\left(k,m,p(t,Z)\right)$ *is defined as in* (4.3.29) *in Corollary 4.5.*

Proof. The proofs to (5.3.1) and (5.3.3) are obtained by replacing the filtration $\mathcal{F}_\infty^X$ and mapping $p(Z_t)$ in Theorem 4.1 and Corollary 4.5 with the sigma-algebra $\sigma(Z)$ and mapping $p(t,Z)$ in (5.1), and then follow the same steps as in Theorem 4.1 and Corollary 4.5. Furthermore, (5.3.2) and (5.3.4) are obtained by taking expected value of (5.3.1) and (5.3.3). □

Remark 5.2. **Example with one-factor Gaussian copula.** If we for example consider the one-factor Gaussian copula model where $p(t,Z)$ is given by (10.1.1) and Z is a standard normal random variable, then according to (5.3.2) in Corollary 5.1, we have for $0<x<1$ that

$$\mathbb{P}\left[\bar{N}_t^{(m)}\geq x\right]=\int_{-\infty}^{\infty}H_B^{(\mathrm{LR})}\left(x,m,p(t,z)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz+O(m^{-3/2}) \tag{5.3.5}$$

and (5.3.4) in Corollary 5.1 implies that for any integer k such that $0\leq k\leq m$ we have

$$\mathbb{P}\left[N_t^{(m)}=k\right]=\int_{-\infty}^{\infty}\Delta H_B^{(\mathrm{LR})}\left(k,m,p(t,z)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz+O(m^{-3/2}) \tag{5.3.6}$$

where the mappings $H_B^{(\mathrm{LR})}\left(x,m,p\right)$ and $\Delta H_B^{(\mathrm{LR})}\left(k,m,p\right)$ are defined as in Corollary 5.1. Note that the integrals in (5.3.5) and (5.3.6) are very easy and straithforward to numerically evaluate. Furthermore, for the case $x=p(t,z)$ in (5.3.5) we use exactly the same arguments as in Remark 4.4.

Remark 5.3. **Adopting the saddlepoint to static credit portfolio models** It is easy to see that the technique presented in Section 4 and 5 also can be adapted to static credit portfolio models, such as those presented in Frey & McNeil (2002), Frey & McNeil (2003), Lando (2004), McNeil et al. (2005) and Herbertsson (2018). In such static credit portfolio models, we here focus on so called mixed binomial models which works as follows. Let Z be a random variable (discrete or continuous) and let $p(x)\in[0,1]$ be a function such that the random variable $p(Z)$ is well-defined. Let $X_1,X_2,\dots X_m$ be identically distributed random variables such that $X_i=1$ if obligor i defaults before time T and $X_i=0$ otherwise. Furthermore, *conditional on* Z, the random variables $X_1,X_2,\dots X_m$ are *independent* and each X_i have default probability $p(Z)$, that is

$$p(Z)=\mathbb{P}\left[\,X_i=1\,|\,Z\right]. \tag{5.3.7}$$

The economic intuition behind this randomizing of the default probability $p(Z)$ is that Z should represent some common background variable affecting all obligors in the portfolio. Many static mixed binomial models can be obtained directly from their continuous time factor versions by defining X_i and $p(Z)$ as

$$X_i=1_{\{\tau_i\leq T\}}\quad\text{so that}\quad p(Z)=p(T,Z)=\mathbb{P}\left[\,\tau_i\leq T\,|\,Z\right] \tag{5.3.8}$$

where for example the mapping $p(T,Z)$ can be given by (5.1.1) or (5.2.2) and where we remind that T is fixed. Now, by defining N_m as the *number* of defaults in the portfolio up to time T, that is $N_m=\sum_{i=1}^m X_i$, we can for such mixed binomial models directly apply a version of Corollary 5.1 to find very sharp approximations to the quantities $\mathbb{P}\left[\,N_m\geq x\,|\,Z\right]$ and $\mathbb{P}\left[\,N_m=k\,|\,Z\right]$ and thus $\mathbb{P}\left[N_m\geq x\right]$ and $\mathbb{P}\left[N_m=k\right]$.

6. Extensions to heterogeneous portfolios

In this section we will give a very brief outline of how to expand our saddlepoint method for computing the default distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$ to a heterogeneous portfolio. A related discussion can be found e.g in Section 2 in Papageorgiou & Sircar (2009) where the main idea is to split the heterogeneous portfolio into smaller homogeneous subportfolios. In this paper we use similar ideas and then on each such homogeneous subportfolio apply the results of e.g. Theorem 4.1, Corollary 4.5 or Corollary 5.1. Thus, below we present the main idea borrowed from e.g. Papageorgiou & Sircar (2009), and we begin with apply it to the case where the main portfolio is split in two homogeneous subportfolios. We first show the method for an intensity based setting as outlined in Section 3, and then also present the technique in the case with factor models as given in Section 5. Finally, we briefly discuss the case with more than two subportfolios.

Thus, assume that we can split our portfolio of m entities/obligors into two homogeneous subportfolios containing m_1 and m_2 entities where $m=m_1+m_2$. Then we have that $N_t^{(m)}=N_t^{(m_1)}+N_t^{(m_2)}$ where

$N_t^{(m_i)}$ is the number of defaults in subportfolio i, where $i = 1, 2$. Furthermore, we assume that each subportfolio is homogeneous, but the parameters for the two portfolios are different. We can also assume that underlying processes X_t^i driving all the individual default intensities $\lambda_i(X_t^i)$, which are same for all obligors in subportfolio i, are either correlated, or independent. However, for ease of exposure of the main ideas, we will assume the same process X_t for both portfolios (which is the same assumption as done in Section 2 of Papageorgiou & Sircar (2009)), that is $X_t^1 = X_t^2 = X_t$ so that all the individual default intensities in portfolio i are the same and given by $\lambda_i(X_t)$. Thus, the default times in both portfolios $i = 1, 2$ are conditional independent given the the filtration $\mathcal{F}_\infty^X$, using the same notation as in Section 3. Then, for any integer k between 1 and m, we have that

$$\begin{aligned}\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] &= \sum_{j=0}^{k} \mathbb{P}\left[N_t^{(m_1)} = j, N_t^{(m_2)} = k-j \,\middle|\, \mathcal{F}_\infty^X\right] \\ &= \sum_{j=0}^{k} \mathbb{P}\left[N_t^{(m_1)} = j \,\middle|\, \mathcal{F}_\infty^X\right] \mathbb{P}\left[N_t^{(m_2)} = k-j \,\middle|\, \mathcal{F}_\infty^X\right]\end{aligned} \tag{6.1}$$

where the second equality in (6.1) follows from the conditional independence. Next we can apply Corollary 4.5 to each of the terms $\mathbb{P}\left[N_t^{(m_i)} = j \,\middle|\, \mathcal{F}_\infty^X\right]$ for $i = 1, 2$ and then immediately get that

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, \mathcal{F}_\infty^X\right] = \sum_{j=0}^{k} \Delta H_B^{(\mathrm{LR})}\left(j, m_1, p(Z_t^1)\right) \Delta H_B^{(\mathrm{LR})}\left(k-j, m_2, p(Z_t^2)\right) + O\left(\max\left(m_1^{-3/2}, m_2^{-3/2}\right)\right) \tag{6.2}$$

where the mapping $\Delta H_B^{(\mathrm{LR})}(k, m, p)$ is defined as (4.3.29) in Corollary 4.5, and where $p(Z_t^i)$ is the conditional default probability at time t given $\mathcal{F}_\infty^X$, for each obligor in portfolio i and the factor Z_t^i defined as in Proposition 3.1, that is

$$Z_t^i = \int_0^t \lambda_i(X_u) du \quad \text{and} \quad p(Z_t^i) = 1 - e^{-Z_t^i} = \mathbb{P}\left[\tau^{(i)} \le t \,\middle|\, \mathcal{F}_\infty^X\right] \quad \text{for } i = 1, 2$$

where $\tau^{(i)}$ has the same distribution as the default times for all obligors in portfolio i, which are exchangeable by construction. To find the unconditional distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ we integrate (6.2) over the bivariate factors (Z_t^1, Z_t^2) which need the joint density $f_{Z_t^1, Z_t^2}(z_1, z_2)$ and there are various of ways to find this bivariata density, see e.g. in Section 2 of Papageorgiou & Sircar (2009).

The ideas that lead to (6.2) can also be applied to factor copula models. For example, considering a Gaussian one-factor model, for the same portfolio structure of two subportfolio as in the example above, and letting Z be the same factor in each portfolio $i = 1, 2$, then, by following the same arguments that gave (6.2), we obtain

$$\mathbb{P}\left[N_t^{(m)} = k \,\middle|\, Z\right] = \sum_{j=0}^{k} \Delta H_B^{(\mathrm{LR})}\left(j, m_1, p_1(t, Z)\right) \Delta H_B^{(\mathrm{LR})}\left(k-j, m_2, p_2(t, Z)\right) + O\left(\max\left(m_1^{-3/2}, m_2^{-3/2}\right)\right) \tag{6.3}$$

with

$$p_i(t, Z) = \Phi\left(\frac{\Phi^{-1}\left(F_i(t)\right) - \sqrt{\rho_i} Z}{\sqrt{1-\rho}}\right) \quad \text{for } i = 1, 2 \tag{6.4}$$

where $F_i(t)$ and ρ_i are the individual default probabilities at time t and correlation parameters in each of the portfolios $i = 1, 2$ and $p_i(t, Z) = \mathbb{P}\left[\tau^{(i)} \le t \,\middle|\, Z\right]$ where $\tau^{(i)}$ has the same distribution as the default times for all obligors in portfolio i, which are exchangeable by construction. Furthermore, in the copula factor-model, the unconditional distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ will be much more easier obtained compared with the intensity based setting given by (6.2), since we simply integrate over Z in (6.3) and then get

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \sum_{j=0}^{k} \mathbb{E}\left[\Delta H_B^{(\mathrm{LR})}\left(j, m_1, p_1(t, Z)\right) \Delta H_B^{(\mathrm{LR})}\left(k-j, m_2, p_2(t, Z)\right)\right] + O\left(\max\left(m_1^{-3/2}, m_2^{-3/2}\right)\right) \tag{6.5}$$

where the expected value $\mathbb{E}\left[\Delta H_B^{(\mathrm{LR})}(j, m_1, p_1(t,Z))\,\Delta H_B^{(\mathrm{LR})}(k-j, m_2, p_2(t,Z))\right]$ is given by

$$\begin{aligned}&\mathbb{E}\left[\Delta H_B^{(\mathrm{LR})}(j, m_1, p_1(t,Z))\,\Delta H_B^{(\mathrm{LR})}(k-j, m_2, p_2(t,Z))\right]\\ &\qquad=\int_0^\infty \Delta H_B^{(\mathrm{LR})}(j, m_1, p_1(t,z))\,\Delta H_B^{(\mathrm{LR})}(k-j, m_2, p_2(t,z))\,\varphi(z)\,dz\end{aligned} \tag{6.6}$$

and $\varphi(z)$ as usual denotes the density to a standard normal random variable, that is $\varphi(z)=\frac{e^{-z^2/2}}{\sqrt{2\pi}}$. Again, we note that the right hand side of (6.6) is easy and parsimonious to evaluate with numerical quadrature procedures.

In the above examples, both for the intensity based model, and the copula factor based model, we assumed two subportfolios where $m=m_1+m_2$ and $N_t^{(m)}=N_t^{(m_1)}+N_t^{(m_2)}$ with $N_t^{(m_i)}$ being the number of defaults in subportfolio i for $i=1,2$. It is easy to see how to extend the formulas in (6.2) and (6.3)-(6.6) to a general case with say, d different subportfolios, $d\leq m$, where each subportfolio is homogeneous (i.e. the obligors in each subportfolio are exchangeable) so that

$$N_t^{(m)}=N_t^{(m_1)}+N_t^{(m_2)}+\ldots+N_t^{(m_d)}\quad\text{ and }\quad m=m_1+m_2+\ldots+m_d\,.$$

In the intensity based case, the unconditional distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$ will in such a setting need the joint distribution of the factors $(Z_t^1,\ldots,Z_t^d)$, which in some cases/models can be numerically challenging when $d>2$. However, in the one-factor Gaussian copula case it will still be straightforward to compute $\mathbb{P}\left[N_t^{(m)}=k\right]$ for any $d\leq m$, since the integration will be done over a one-dimensional integral. For example, if $d=3$ then (6.5) can be extended to

$$\begin{aligned}&\mathbb{P}\left[N_t^{(m)}=k\right]\\ &=\sum_{j_1=0}^{k}\sum_{j_2=0}^{k-j_1}\mathbb{E}\left[\Delta H_B^{(\mathrm{LR})}(j_1, m_1, p_1(t,Z))\,\Delta H_B^{(\mathrm{LR})}(j_2, m_2, p_2(t,Z))\,\Delta H_B^{(\mathrm{LR})}(k-j_1-j_2, m_3, p_3(t,Z))\right]\\ &\quad+O\left(\max\left(m_1^{-3/2}, m_2^{-3/2}, m_3^{-3/2}\right)\right)\end{aligned} \tag{6.7}$$

where each expected value in the dubble-sum in (6.7) is evaluated in the same way as in Equation (6.6).

7. Applications to equity risk management: Value-at-Risk for large homogeneous stock portfolios with jumps at exogenous defaults

There exists a huge amount of applications where the distributions $\mathbb{P}\left[N_t^{(m)}\geq x\right]$ and $\mathbb{P}\left[N_t^{(m)}=k\right]$ are used, particular in credit risk, for example risk management of credit portfolios done under the real probability measure, but also for credit portfolio derivative pricing. For more discussions of applications of $\mathbb{P}\left[N_t^{(m)}\geq x\right]$ and $\mathbb{P}\left[N_t^{(m)}=k\right]$ in credit risk, see for example in Herbertsson (2023*b*). However, in this paper we will focus on applications of $\mathbb{P}\left[N_t^{(m)}=k\right]$ in equity risk management in a stock price model developed in Herbertsson (2023*a*) where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities, for example corporates or sovereign states. By "exogenous" we here mean that the entities, for example companies, will not be represented in the stock portfolio, that is stocks issued by the defaultable corporates are not present in the stock portfolio in our studies. The default times can come from any type of credit portfolio model. In this setting Herbertsson (2023*a*) derive computational tractable formulas to several stock-related quantizes, for example the loss distributions of equity portfolios and apply it to risk management computations such as Value-at-Risk of portfolios. For the stock portfolio case Herbertsson (2023*a*) considers both small-time expansions of the loss-distribution to a heterogeneous portfolio via a linearization of the loss, but also for general time points when the stock portfolio is large and homogeneous, where Herbertsson (2023*a*) utilize a conditional version of the law of large numbers for a homogeneous stock portfolio. In the numerical examples in Herbertsson (2023*a*), the credit portfolio models used are homogeneous and conditional independent and therefore all credit related computations for $\mathbb{P}\left[N_t^{(m)}=k\right]$ done in Herbertsson (2023*a*) heavily rely on efficient numerical methods developed in this paper for computing the distribution of number of defaults

$\mathbb{P}\left[N_t^{(m)} = k\right]$ among the defaultable entities creating the jumps in the stock prices. For more numerical details we refer to Herbertsson (2023*a*).

In this paper we will perform some complementary numerical studies of the stock price model developed in Herbertsson (2023*a*), which are not present in Herbertsson (2023*a*). More specific, in Section 10 we will present numerical results to Value-at-Risk for a large stock portfolio, as function of the default correlation parameter in the one-factor Gaussian copula model, at different time points and for different confidence levels. Such studies are not done in Herbertsson (2023*a*), and are directly dependent on efficient and fast computations of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$.

In Subsection 7.1 we first give a brief introduction to the portfolio stock price model developed in Herbertsson (2023*a*) where all stock prices can jump at default times belonging to an exogenous group of defaultable entities. Then, in Subsection 7.2 we define the loss process for the stock portfolio and present formulas for the loss distribution without proofs, where full proofs of the expressions can be found in Herbertsson (2023*a*). More specific, in Subsection 7.2 we focus on large homogeneous stock portfolios, and present convenient expression for the distribution of the portfolio loss in such settings, which are based on the large portfolio approximations. Full proofs to the derivations can be found in Herbertsson (2023*a*) and Herbertsson (2023*a*) also derives the stock portfolio loss distribution for heterogeneous portfolios, which is not considered in this paper.

7.1. **The stock price model.** In this subsection we give a brief introduction to the portfolio stock price model of Herbertsson (2023*a*) where each stock price in the equity portfolio can jump at default times belonging to an exogenous group of defaultable entities. The dynamics of the stock prices are done under the real (physical) probability measure $\mathbb{P}$ that will be used throughout the rest of this section. We start with the following definition of the stock prices, borrowed from Herbertsson (2023*a*).

Definition 7.1. Consider a group of m defaultable entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ with individual default times $\tau_1, \tau_2 \ldots, \tau_m$ and let $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Let the companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ be J different exchangeable entities which do not belong to the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$ and let $S_{t,1}, \ldots, S_{t,J}$ denote the stock prices of the companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ at time t under the real probability measure $\mathbb{P}$. Then, for each entity $\mathbf{A}_j$ we define the stock price $S_{t,j}$ as

$$S_{t,j} = S_{0,j} \exp\left(\left(\mu_j - \frac{1}{2}\sigma_j^2\right) t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2} W_{t,j}\right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j}\right) \tag{7.1.1}$$

where $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ are $J+1$ independent Brownian motions and $\rho_{S,j} \in [-1,1]$ are constants. Furthermore, for each $j = 1, 2, \ldots, J$ the m random variables $U_{1,j}, \ldots, U_{m,j}$ are an i.i.d sequence distributed as

$$U_{n,j} \overset{d}{=} \text{Exp}(\eta_j) \quad \text{with} \quad \mathbb{E}\left[U_{n,j}\right] = \frac{1}{\eta_j} \tag{7.1.2}$$

where $U_{1,j}, \ldots, U_{m,j}$ are independent of the processes $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ and also independent of the default times $\tau_1, \tau_2 \ldots, \tau_m$. Furthermore, for each company $\mathbf{A}_j$ the parameters $\sigma_j > 0$ and μ_j are the volatility and drift, same as in the one-dimensional defined in Herbertsson (2023*a*).

We next make some remarks connected to Definition 7.1.

Remark 7.2. We remark that the default times $\tau_1, \tau_2 \ldots, \tau_m$ in Definition 7.1 can come from any credit portfolio model as long as the jumps $\tilde{V}_1, \ldots, \tilde{V}_m$ in the stock prices at the default times $\tau_1, \tau_2 \ldots, \tau_m$ are independent of these defaults and also independent of the Brownian motion. We can for example work with heterogeneous or homogeneous copula based models studied in e.g. Li (2000), Gregory & Laurent (2005), Gregory & Laurent (2003), Andersen & Sidenius (2004), Crépey et al. (2013), Burtschell et al. (2009), Hofert & Scherer (2011) or heterogeneous or homogeneous conditional independent intensity based models such as in Bielecki, Cousin, Crépey & Herbertsson (2014*b*), Bielecki et al. (2014*c*) and Bielecki, Cousin, Crépey & Herbertsson (2014*a*) as well as heterogeneous or homogeneous contagion models studied in e.g. Herbertsson (2005), Herbertsson (2007), Herbertsson & Rootzén (2008), Herbertsson (2008*b*), Herbertsson (2008*a*), Herbertsson (2011), Cont, Deguest & Kan (2010), Cont & Kan (2011), Laurent, Cousin & Fermanian (2011), Frey & Backhaus (2008) and Frey & Backhaus (2010).

Remark 7.3. If we let $\tilde{U}_{1,j}, \ldots, \tilde{U}_{m,j}$ be an i.i.d sequence with same distribution as $U_{1,j}, \ldots, U_{m,j}$ then the jump term $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in (7.1.1) can be replaced by the more intuitive expression $\sum_{i=1}^{m} \tilde{U}_{i,j} 1_{\{\tau_i \leq t\}}$, since $\sum_{n=1}^{N_t^{(m)}} U_{n,j} \stackrel{d}{=} \sum_{i=1}^{m} \tilde{U}_{i,j} 1_{\{\tau_i \leq t\}}$.

Remark 7.4. Note that $\rho_{S,j} \in [-1, 1]$ and unless explicitly stated, we will throughout this paper always assume that at least one company $\mathbf{A}_j$ has a correlation such that $\rho_{S,j} \neq -1, 1$ so that $\rho_{S,j} \in (-1, 1)$.

Remark 7.5. In the case when there are no jump at the defaults in Definition 7.1, i.e when $U_n = 0$ for all n, then $S_{t,j} = S_{t,j}^{(\mathrm{BS})}$ for all companies $\mathbf{A}_j$ with $S_{t,j}^{(\mathrm{BS})}$ given by

$$S_{t,j}^{(\mathrm{BS})} = S_{0,j} \exp\left(\left(\mu_j - \frac{1}{2}\sigma_j^2\right) t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j}\right)\right) \tag{7.1.3}$$

where $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ are $J + 1$ independent Brownian motions and the rest of the notation is same as in Definition 7.1. Note that $S_{t,1}^{(\mathrm{BS})}, \ldots, S_{t,J}^{(\mathrm{BS})}$ will under (7.1.3) still be correlated via the factor process $W_{t,0}$ and recall that $\rho_S W_{t,0} + \sqrt{1 - \rho_S^2} W_{t,j}$ is a Brownian motion for each stock price $S_{t,j}^{(\mathrm{BS})}$.

Remark 7.6. **The stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ are correlated and have simultaneous jumps.** Since $W_{t,0}$ and $W_{t,j}$ are independent Brownian motions for each j and $\rho_{S,j} \in [-1, 1]$, then from standard probability theory we know that $\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j}$ used in (7.1.1) is also a Brownian motion. Hence, one can prove (see e.g. Corollary 2.11 in Herbertsson (2023*a*)) that the dynamics of the stock price $S_{t,j}$ for each firm $\mathbf{A}_j$ satisfies

$$dS_{t,j} = S_{t-,j} dY_{t,j} \tag{7.1.4}$$

where $Y_{t,j}$ is given by

$$Y_{t,j} = \mu_j t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1 - \rho_{S,j}^2} W_{t,j}\right) + \sum_{n=1}^{N_t^{(m)}} \left(e^{-U_{n,j}} - 1\right) . \tag{7.1.5}$$

For a full proof of the connection between Definition 7.1 and the dynamics (7.1.4) - (7.1.5), see e.g. in Herbertsson (2023*a*). Further, from the construction of $S_{t,j}$ in (7.1.1) and $U_{n,j}$ in (7.1.2), stated in Definition 7.1, the stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ will be "correlated" via the factor process $W_{t,0}$ when $\rho_{S,j} \neq 0$, and also "correlated" via the default counting process $N_t^{(m)}$ for the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$. In particular, all stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ will have a jump at the default times $\tau_1, \tau_2 \ldots, \tau_m$, where the relative jumps of $S_{t,j}$ will be different almost surely under $\mathbb{P}$, although have same distribution, given by (7.1.2). Expressions for the expected value, conditional expected value, density and distribution for each stock price $S_{t,j}$ is found in Herbertsson (2023*a*).

From Theorem 2.12 in Herbertsson (2023*a*) we know that

$$\mathbb{E}\left[S_t\right] = S_0 e^{\mu t} \mathbb{E}\left[\left(\frac{\eta}{\eta + 1}\right)^{N_t^{(m)}}\right] = S_0 e^{\mu t} \sum_{k=0}^{m} \left(\frac{\eta}{\eta + 1}\right)^k \mathbb{P}\left[N_t^{(m)} = k\right] \tag{7.1.6}$$

and the relation (7.1.6) can be used when calibration the jump parameter η as will be discussed more in Section 10.

Consider a weighted stock portfolio consisting of $w_1, w_2, \ldots, w_J$ stocks chosen for our portfolio at time $t = 0$, where the stocks are issued by the J companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ with stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ that satisfy Definition 7.1. Then we define the portfolio value V_t as

$$V_t = \sum_{j=1}^{J} w_j S_{t,j} . \tag{7.1.7}$$

Next, we, define the portfolio loss process $L_t^{(V)}$ for a general portfolio V_t at time t with reference to the starting time 0, as

$$L_t^{(V)} = -\left(V_t - V_0\right) \tag{7.1.8}$$

where we note that a gain implies that the loss $L_t^{(V)}$ is negative. We are interested to compute Value-at-Risk for $L_t^{(V)}$ in our model given by Definition 7.1, that is, we want to compute

$$\text{VaR}_\alpha\left(L_t^{(V)}\right) = \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[L_t^{(V)} > y\right] \leq 1-\alpha\right\} = \inf\left\{y \in \mathbb{R} : F_{L_t^{(V)}}(y) \geq \alpha\right\} \tag{7.1.9}$$

where $F_{L_t^{(V)}}(x)$ is the distribution of $L_t^{(V)}$ and α is the confidence level. Unfortunately, finding analytical or semi-analytical expressions to $F_{L_t^{(V)}}(x)$ is a challenging task. However, assuming that exponents in $S_{t,j}$ will be small for small t, then Herbertsson (2023a) uses a first order Taylor expansion of the terms $S_{t,j}$ in a so called equally value-weighted portfolio together with the assumption $\eta = \eta_1 = \eta_2 = \ldots = \eta_J$, to find semi-analytical approximations to the loss process $L_t^{(V)}$, see Theorem 3.8 in Herbertsson (2023a). Furthermore, Herbertsson (2023a) use this small-time expansions of the loss-distribution to a heterogeneous portfolio via a linearization of the loss, in order to numerically compute the time evolution of Value-at-Risk (i.e. VaR as function of time) for stock portfolios, for a 20-day period with one-day steps, in a setting where the jumps in the stock prices are at default times which are generated by one-factor Gaussian copula model.

7.2. **Approximation formulas to the loss distributions for large homogeneous stock portfolios with jumps at exogenous defaults.** For larger time points t, the linear approximations to the stock portfolio discussed in the previous subsection will fail. For example, the linearized loss may produce VaR-values that are bigger than V_0 which is impossible since by construction it will hold that $L_t^{(V)} \leq V_0$ almost surely for all $t \geq 0$ under the real probability measure $\mathbb{P}$. However, in certain cases we can still find highly analytical approximation formulas for the loss distribution $\mathbb{P}\left[L_t^{(V)} \leq x\right]$ at any time point t and where the loss will never exceed V_0, as will be seen in the next subsection. More specific, if we assume that the stock prices $S_{t,j}$ satisfy

$$S_{0,j} = S_0, \quad \mu_j = \mu, \quad \sigma_j = \sigma, \quad \text{and} \quad \rho_{S,j} = \rho_S \; \eta = \eta_j \quad \text{for all firms} \quad \mathbf{A}_1, \ldots, \mathbf{A}_J \tag{7.2.1}$$

so that the stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ become exchangeable, and if the number of stocks J in the portfolio are "large", then Herbertsson (2023a) derive approximation formulas for the loss distribution $\mathbb{P}\left[L_t^{(V)} \leq x\right]$ which will work for arbitrary time points t, that is both for large and small time points t and which will also guarantee that portfolio loss always will be smaller than V_0 almost surely for all $t \geq 0$ under the measure $\mathbb{P}$.

Hence, in the rest of this section we will make two assumptions. First we assume that (7.2.1) holds together with Definition 7.1 under the real probability measure $\mathbb{P}$, with equal portfolio weights w_j for all companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ in the portfolio V_t. Our second assumption is that the number of stocks J in the portfolio are "large". Since the stock portfolio is equally weighted, and we are primary interested in Value-at-Risk calculation of the portfolio, then due to linearity of VaR we can without loss of generality let $w_j = 1$ for each stock in the portfolio and thus define the portfolio value as $V_t = \sum_{j=1}^J S_{t,j}$. Due to the condition (7.2.1) the portfolio V_t given by will then be an equally value-weighted portfolio, see also in Herbertsson (2023a).

Remark 7.7. **Homogenization of a heterogeneous stock portfolio:** Assuming a completely homogeneous stock portfolio so that the parameters for each stock are the same is of course an unrealistic feature. Consider a heterogeneous stock portfolio with stocks defined as in Definition 7.1 and portfolio value $\hat{V}_t$ and define S_0, μ, σ and ρ_S as the corresponding sample means of the parameters in this portfolio, that is

$$S_0 = \frac{1}{J}\sum_{j=1}^J S_{0,j} \quad \mu = \frac{1}{J}\sum_{j=1}^J \mu_j \quad \sigma = \frac{1}{J}\sum_{j=1}^J \sigma_j \quad \text{and} \quad \rho_S = \frac{1}{J}\sum_{j=1}^J \rho_{S,j} \; . \tag{7.2.2}$$

Next create a homogeneous stock portfolio as in (7.2.1) with parameters S_0, μ, σ and ρ_S given by (7.2.2) and portfolio value V_t and where $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$, $N_t^{(m)}$ and $U_{i,j}$ are the same as in the heterogeneous portfolio. For such homogeneous portfolios Hofmann & Platen (2000) as well as Guan, Xiaoqing & Chong (2003) proves that the value process $\hat{V}_t$ for a large heterogeneous stock portfolio can be approximated arbitrary well by V_t in L_1-sense as $J \to \infty$. Hofmann & Platen (2000) proves the result for portfolios with only diffusions while Guan et al. (2003) extends the proof to the case where the stocks also can jump due to Poisson processes. In view of the results of e.g. Hofmann & Platen (2000) and Guan et al. (2003) it

is therefore still relevant to consider homogeneous stock portfolios in particular if these portfolios comes from doing a homogenization of a heterogeneous stock portfolio as in (7.2.2).

Given the assumption that (7.2.1) is satisfied, Herbertsson (2023*a*) proves the following theorem.

Theorem 7.8. *Let $S_{t,1},\ldots,S_{t,J}$ be stock prices defined as in Definition 7.1 which satisfies* (7.2.1) *under the real probability measure $\mathbb{P}$. Then, with notation as above, we have*

$$\lim_{J\to\infty}\frac{1}{J}\sum_{j=1}^{J}S_{t,j}=S_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\ \text{a.s. under}\ \mathbb{P}\left[\cdot\,|\,W_{t,0},N_t^{(m)}\right] \tag{7.2.3}$$

and

$$\lim_{J\to\infty}\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J}S_{t,j}\le x\right]=\mathbb{P}\left[S_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\le x\right]. \tag{7.2.4}$$

Furthermore, for large J we have

$$\mathbb{P}\left[L_t^{(V)}\le x\right]\approx\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right)\le x\right]\ \text{for large}\ J \tag{7.2.5}$$

and if $\rho_S\neq 0$ then for $x\le JS_0=V_0$ it holds that

$$\mathbb{P}\left[L_t^{(V)}\le x\right]\approx 1-\sum_{k=0}^{m}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)\mathbb{P}\left[N_t^{(m)}=k\right]\ \text{for large}\ J \tag{7.2.6}$$

where $\Phi(x)$ is the distribution function to a standard normal random variable.

Note that loss distribution formula in (7.2.6) in Theorem 7.8 requires efficient and quick methods of computing the number of default distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$. In Section XX we will apply the formulas in Theorem 7.8 together with the efficient methods for $\mathbb{P}\left[N_t^{(m)}=k\right]$ derived in Corollary 4.5 and Corollary 5.1.

For $\rho_S\neq 0$, define $F^{\text{LPA}}_{L_t^{(V)}}(x)$ as

$$F^{\text{LPA}}_{L_t^{(V)}}(x)=1-\sum_{k=0}^{m}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)\mathbb{P}\left[N_t^{(m)}=k\right]. \tag{7.2.7}$$

Then, if $\rho_S\neq 0$, the large portfolio approximation formula (7.2.6) in Theorem 7.8 implies that

$$\mathbb{P}\left[L_t^{(V)}\le x\right]\approx F^{\text{LPA}}_{L_t^{(V)}}(x)\quad\text{for large}\ J. \tag{7.2.8}$$

Let $\text{VaR}_\alpha\left(L_t^{(V)}\right)$ defined as in (7.1.9) be the Value-at-Risk for the stock portfolio loss $L_t^{(V)}$ with confidence level α. By using the large portfolio approximation formula (7.2.6) in Theorem 7.8, that is, relation (7.2.8), we can for large J find an approximation to $\text{VaR}_\alpha\left(L_t^{(V)}\right)$ which then is given as the unique solution x^* to the equation $F^{\text{LPA}}_{L_t^{(V)}}(x^*)=\alpha$, that is

$$\text{VaR}_\alpha\left(L_t^{(V)}\right)\approx(F^{-1})^{\text{LPA}}_{L_t^{(V)}}(\alpha)\quad\text{for large}\ J \tag{7.2.9}$$

where $(F^{-1})^{\text{LPA}}_{L_t^{(V)}}(x)$ denotes the inverse function to the function $F^{\text{LPA}}_{L_t^{(V)}}(x)$ defined in (7.2.7). Since $F^{\text{LPA}}_{L_t^{(V)}}(x)=1$ for $x>V_0=JS_0$ we see that (7.2.9) can never produce a VaR-value bigger than V_0, contrary to the linearized portfolio loss VaR-values.

In the case when there are no jumps in the stock prices at the defaults of the exogenous group of defaultable entities in Definition 7.1, i.e when "$\eta=\infty$" so that $U_{n,j}=0$ for all pairs n,j and thus

$S_{t,j} = S_{t,j}^{(\mathrm{BS})}$ for all companies $\mathbf{A}_j$ where $S_{t,j}^{(\mathrm{BS})}$ is given by (7.1.3) in Remark 7.5 and if $\rho_S \neq 0$, then (7.2.3) in Theorem 7.8 will reduce to

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^{J} S_{t,j} = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right) \text{ a.s. under the random measure } \mathbb{P}\left[\,\cdot\,|\,W_{t,0}\right]. \tag{7.2.10}$$

Hence, from (7.2.10) and using the same arguments as in Theorem 7.8 we then have that

$$\mathbb{P}\left[L_t^{(V)} \leq x\right] \approx \mathbb{P}\left[JS_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\right) \leq x\right] \quad \text{for large } J. \tag{7.2.11}$$

We also note that the right hand side in (7.2.10) is on the exact same form as the stock price $S_t^{(\mathrm{BS})}$ in the Black-Scholes model for a single stock, under the real probability measure $\mathbb{P}$ with the volatility $\sigma\rho_S$. Hence, for large J, the loss process $L_t^{(V)}$ will for the case when $U_{n,j} = 0$ for all n, j behave as the loss process for one single stock which follows the Black-Scholes dynamics with volatility $\sigma\rho_S$, drift μ and initial value JS_0. In the Black-Scholes model it is possible to find an analytical expression for Value-at-Risk of the loss-process, see for example in Herbertsson (2023*a*), and this observation together with the large portfolio approximation in (7.2.11) implies that in the case with no jumps in the stock prices we get that

$$\mathrm{VaR}_\alpha\left(L_t^{(V)}\right) \approx JS_0\left(1 - \exp\left(\sigma\rho_S\sqrt{t}\Phi^{-1}\left(1-\alpha\right) + \left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t\right)\right) \quad \text{for large } J. \tag{7.2.12}$$

In our numerical studies in Section 10 we will use the "Black-Scholes" LPA VaR formula in (7.2.12) as benchmark for the VaR-formulas obtained when using the LPA loss distribution (7.2.6) in Theorem 7.8 when the stock prices have jumps and are exchangeable.

In Section 10 we will also perform some complementary numerical studies of the stock price model developed in Herbertsson (2023*a*), which are not present in Herbertsson (2023*a*). For example, in Section 10 we will present numerical results to Value-at-Risk for a large stock portfolio, as function of the default correlation parameter in the one-factor Gaussian copula model, at different time points and for different confidence levels. Such studies are not done in Herbertsson (2023*a*), and are directly dependent on efficient and fast computations of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$. In all our VaR-computations we will apply the formulas in Theorem 7.8 together with the efficient methods for $\mathbb{P}\left[N_t^{(m)} = k\right]$ derived in Corollary 5.1.

8. Numerical examples when the default intensity is a CIR-process

The results for the unconditional probability expressions in Theorem 4.1 and Corollary 4.5 implies that these formulas are only relevant in practice if we are able to find analytical expressions of the density $f_{Z_t}(z)$ to the random variable Z_t defined as in (3.2.1). For many important intensity based credit models this is possible via Fourier methods or the saddlepoint approach due to the fact that we often have highly analytical expression for the moment generating function to the random variable Z_t in terms of the parameters describing the individual default intensity $\lambda(X_t)$. Examples of such settings are the CIR modell described in Subsection 3.1.1. In this section we provide numerical examples of Theorem 4.1 and Corollary 4.5. First, in Subsection 8.1 we present the characteristic function to the integrated CIR process Z_t and outline how we retrieve the density $f_{Z_t}(z)$ numerically. We then also display the obtained $f_{Z_t}(z)$ for some different time points. Next, in Subsection 8.2 we use the densities $f_{Z_t}(z)$ to compute the quantities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ and $\mathbb{P}\left[N_t^{(m)} = k\right]$ via Theorem 4.1 and Corollary 4.5.

8.1. Characteristic functions to the integrated CIR-process and how to find their numerical values.

Let $\lambda_t = \lambda\left(X_t\right) = X_t$ be a Cox-Ingersoll-Ross process (CIR-process) as presented in Subsection (3.1.1), that is

$$d\lambda_t = a\left(\mu - \lambda_t\right)dt + \sigma\sqrt{\lambda_t}dW_t \tag{8.1.1}$$

where W_t is a Brownian motion under the physical probability measure $\mathbb{P}$. Then it is possible to find convenient analytical expressions for the Laplace transform, or equivalently the momentgenerating function $M_{Z_t}(s)$ (recall that the Laplace transform, is obtained from $M_{Z_t}(s)$ by replacing s with $-u$ where $u > 0$) to $Z_t = \int_0^t \lambda(X_u)du$ where $X_t = \lambda_t$ is a CIR-process, see e.g. Theorem 9.6.4 on p.273 in Elliott & Kopp (2005) or Proposition 6.2.4 on p.130 in Lamberton & Lapeyre (1996), see also Lemma 3.1. on pp. 1367 in Bielecki et al. (2014*c*). The corresponding characteristic function $\varphi_{Z_t}(u) = \mathbb{E}\left[e^{iuZ_t}\right]$ is obtained

from Laplace transform $M_{Z_t}(-s)$ by letting $s = iu$ where i is the imaginary unit, that is $i^2 = -1$. Hence, $\varphi_{Z_t}(u) = K_{Z_t}(iu)$. Here we state the expressions for $\varphi_{Z_t}(u)$ since we need it for our numerical implementations. Let $Z_t = \int_0^t \lambda(X_u)du$ where $X_t = \lambda_t$ is a CIR-process as in (8.1.1). Then, $\varphi_{Z_t}(u) = \mathbb{E}\left[e^{iuZ_t}\right]$ is given by

$$\varphi_{Z_t}(u) = \mathbb{E}\left[e^{iuZ_t}\right] = e^{a\mu\psi_t(u)-\lambda_0\nu_t(u)} \tag{8.1.2}$$

where

$$\psi_t(u) = \frac{2}{\sigma^2}\ln\left(\frac{2\gamma(u)e^{t(a+\gamma(u))/2}}{\gamma(u)-a+e^{t\gamma(u)}(\gamma(u)+a)}\right) \tag{8.1.3}$$

and

$$\nu_t(u) = \frac{e^{t\gamma(u)}\left(\gamma(u)-a\right) - 2iu\left(e^{t\gamma(u)}-1\right)}{\gamma(u)-a+e^{t\gamma(u)}(\gamma(u)+a)} \tag{8.1.4}$$

with

$$\gamma(u) = \sqrt{a^2 - 2iu\sigma^2} \tag{8.1.5}$$

In Theorem 9.6.4 on p.273 in Elliott & Kopp (2005) and Proposition 6.2.4 on p.130 in Lamberton & Lapeyre (1996) the authors state the Laplace transform of these expression, and not the moment generation functions. Hence, the expression in (8.1.2) is obtained by replacing the arguments in Elliott & Kopp (2005) and Lamberton & Lapeyre (1996) with $-iu$, as opposed to iu for the momgentgenerating functions.

Next, the density $f_{Z_t}(z)$ to the random variable Z_t is obtained from the inversion formula

$$f_{Z_t}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\varphi_{Z_t}(u)\,du \tag{8.1.6}$$

and we also have that

$$\mathbb{P}\left[Z_t \le z\right] = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty}\frac{e^{iuz}\varphi_{Z_t}(-u) - e^{-iuz}\varphi_{Z_t}(u)}{iu}\,du \tag{8.1.7}$$

see e.g. p.481 in in Gil-Pelaez (1951). By differentiating (8.1.7) with respect to z and using rules for complex conjugates etc. one can obtain a slightly more practical version of (8.1.6) given by

$$f_{Z_t}(z) = \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(e^{-iuz}\varphi_{Z_t}(u)\right)\,du\,. \tag{8.1.8}$$

Since $\varphi_{Z_t}(u)$ is given in closed form, which is rather to quick to evaluate numerically, then a good approximiation to $f_{Z_t}(z)$ is obtained by numerical quadrature of (8.1.8), that is,

$$f_{Z_t}(z) \approx \frac{1}{\pi}\sum_{n=1}^{I_K}\mathrm{Re}\left(e^{-iu_nz}\varphi_{Z_t}(u_n)\right)\Delta u_n\,. \tag{8.1.9}$$

where $0 \le u_1 < u_2 < ... < u_{I_K} = K$ is a discritisation of some intervall $[0, K]$ where both K and I_K typically are large numbers and $\Delta u_n = u_{n+1} - u_n$. In our numerical studies we notice that one often has to have quite a large amount of terms in the sum (that is I_K have to be large, and Δu_n quite small), that is, the convergence rate is slow, which is something that has been emphasized in the literature. For example, Fang & Oosterlee (2008) and Feng & Lin (2013) remark that since the integrands in (8.1.8) - (8.1.9) are highly oscillatory, a relatively fine grid in the discrete sum (8.1.9) has to be used in order to obtain a good a accuracy the density $f_{Z_t}(z)$. Fortunately, since the evaluation of the integrand in the CIR-case is quite quick it still possible to obtain a good approximation of $f_{Z_t}(z)$ in reasonable quick time, even if we use a lot of terms in the sum. Once we obtain the values of $f_{Z_t}(z)$ on the mesh $z_1 < z_2 < \ldots z_{J_C}$ which approximates the positive real line $[0, \infty)$, we can also check if it approximately holds that

$$\sum_{n=1}^{J_C} f_{Z_t}(z_n)\Delta z_n \approx 1 \quad \text{and} \quad \sum_{n=1}^{J_C} z_n f_{Z_t}(z_n)\Delta z_n \approx \mathbb{E}\left[Z_t\right] = \mu t + \frac{1}{a}\left(\lambda_0 - \mu\right)\left(1 - e^{-at}\right) \tag{8.1.10}$$

where the closed formula for $\mathbb{E}\left[Z_t\right]$ is obtained by using that $\mathbb{E}\left[Z_t\right] = \mathbb{E}\left[\int_0^t \lambda_s\,ds\right] = \int_0^t \mathbb{E}\left[\lambda_s\right]\,ds$ and then use the expression for $\mathbb{E}\left[\lambda_s\right]$ given e.g. on p.392 in Cox, Ingersoll & Ross (1985) which leads to the formula for $\mathbb{E}\left[Z_t\right]$ in (8.1.10), see also p.89 in Schoutens (2003). Hence, after obtaining our discrete set $\{f_{Z_t}(z_n)\}$ we can check if the approximations in (8.1.10) will hold making a first verification of the Fourier inversion method. We also note that one can compute (8.1.9) by using the fast Fourier transform FFT algorithm

which can speed up the computations, but this will require a constraint on the relationship between the discrete steps in the z-state and frequency space u, which in this paper not used.

8.2. **Numerical examples with a default intensity following a CIR-proces.** In our numerical example we choose a CIR-process with the parameters $a = 0.6, \mu = 0.056, \sigma = 0.18$ and $\lambda_0 = 0.0262$ so that the individual one-year default probability is 0.0329 computed via the expression $1 - \exp(A(1) - B(1)\lambda_0)$ with $A(T)$ and $B(T)$ defined as in (3.1.1.3)-(3.1.1.5), see also in Table 1.

Table 1. The parameters and related quantities for the CIR-process λ_t.

λ_t	$\lambda_0 = 0.0262 \quad a = 0.6 \quad \mu = 0.056 \quad \sigma = 0.18 \quad \mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$

We obtain $f_{Z_t}(z)$ via the Fourier inversion formula (8.1.9) on the range $z \in [0.00005, 0.28]$ and then verify that (8.1.10) holds, before proceeding with our other computations. Figure 3 shows the densities $f_{Z_t}(z)$ to $Z_t = \int_0^t \lambda_u du$ for $t = 4, 5, \ldots, 24$ months and $z \in [0.00005, 0.28]$ where λ_t is a CIR-process with parameters as in Table 1. Furthermore, the densities $f_{Z_t}(z)$ for $t = 1, 2, 3, 4$ are displayed separately in Figure 4 due to the very high values of $f_{Z_t}(z)$ for some z-arguments, where the CIR-parameters are same as in Table 1 and Figure 3. In Figure 5 we show the densities $f_{Z_t}(z)$ for $t = 21, 22, 23, 24$ months where $z \in [0.00005, 0.28]$ and where the CIR-parameters are same as in Figure 3. In Figure 3 - 5 we write t in months, but the actual computations when finding $f_{Z_t}(z)$ are done with t measured in units of years. So for example two, six and 24 months means that t is given by $t = \frac{2}{12}, \frac{6}{12}$ and $t = \frac{24}{12} = 2$ in our computations.

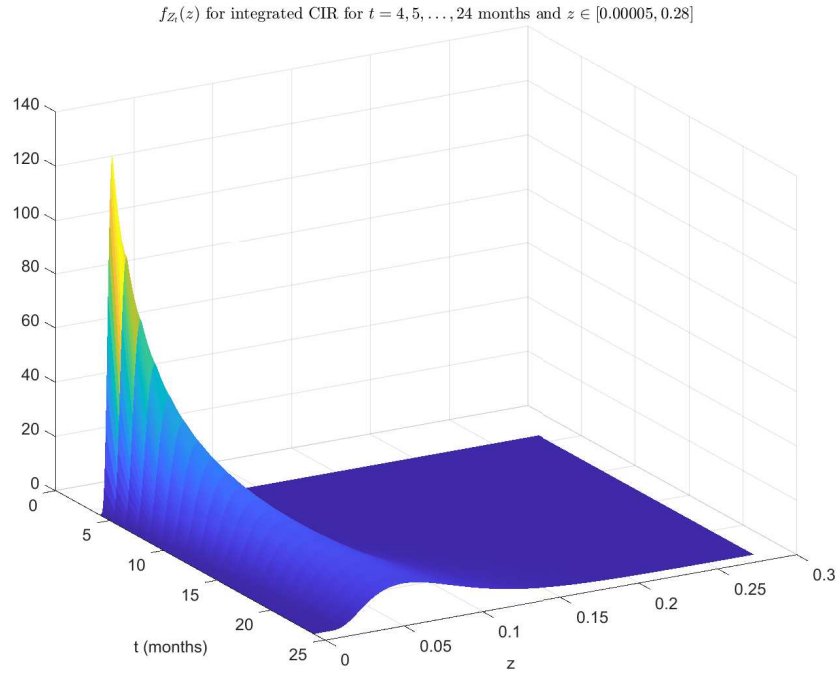


Figure 3. The densities $f_{Z_t}(z)$ to $Z_t = \int_0^t \lambda_u du$ for $t = 4, 5, \ldots, 24$ months where λ_t is a CIR-process with parameters as in Table 1. The densities $f_{Z_t}(z)$ are obtained via the Fourier inversion formula (8.1.8) where $z \in [0.00005, 0.28]$.

Once we are equipped with $f_{Z_t}(z)$ for various time points t, we proceed with computing the quantities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ and $\mathbb{P}\left[N_t^{(m)} = k\right]$ via Theorem 4.1 and Corollary 4.5. First, for $m = 125$, the left panel in Figure 6 displays the time evolution of the tail distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, ..24$ months when individual default times have CIR-intensities with parameters same as in Figure 3.

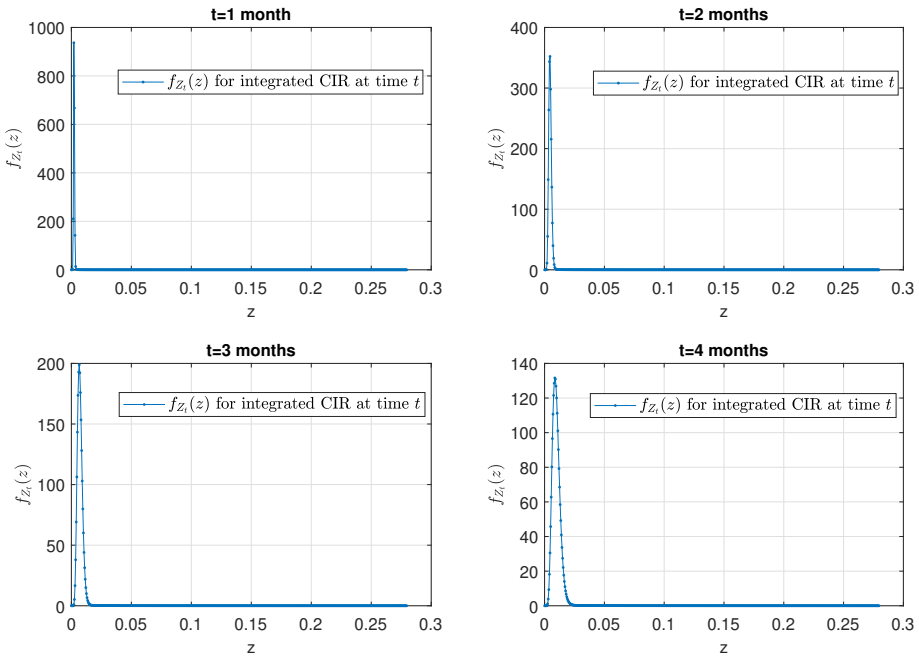


Figure 4. The densities $f_{Z_t}(z)$ to $Z_t = \int_0^t \lambda_u du$ for $t = 1, 2, 3, 4$ months where λ_t is a CIR-process with parameters as in Table 1. The densities $f_{Z_t}(z)$ are obtained via the Fourier inversion formula (8.1.8) where $z \in [0.00005, 0.28]$.

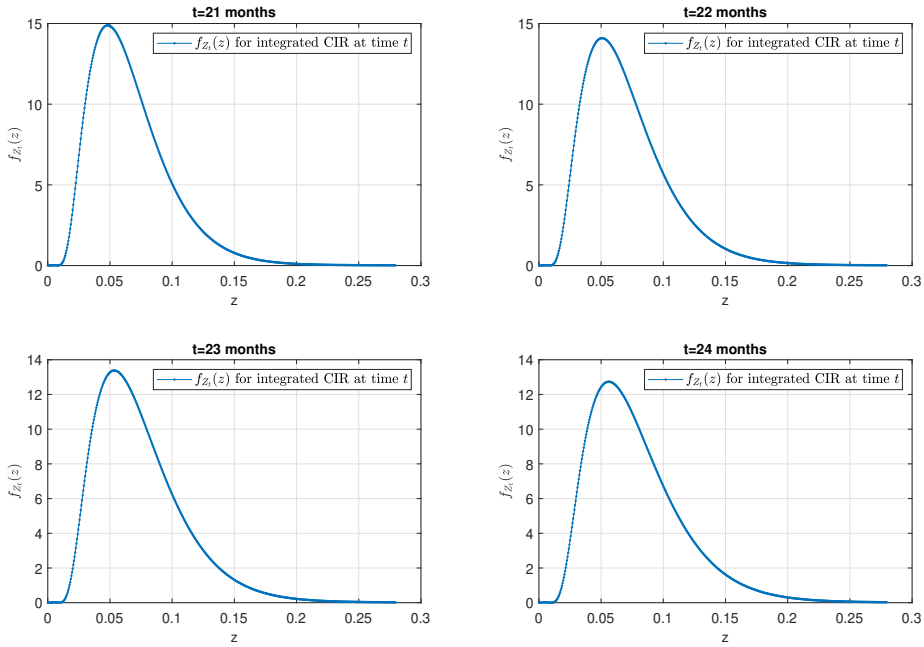


Figure 5. The densities $f_{Z_t}(z)$ to $Z_t = \int_0^t \lambda_u du$ for $t = 21, 22, 23, 24$ months where λ_t is a CIR-process with parameters as in Table 1. The densities $f_{Z_t}(z)$ are obtained via the Fourier inversion formula (8.1.8) where $z \in [0.00005, 0.28]$.

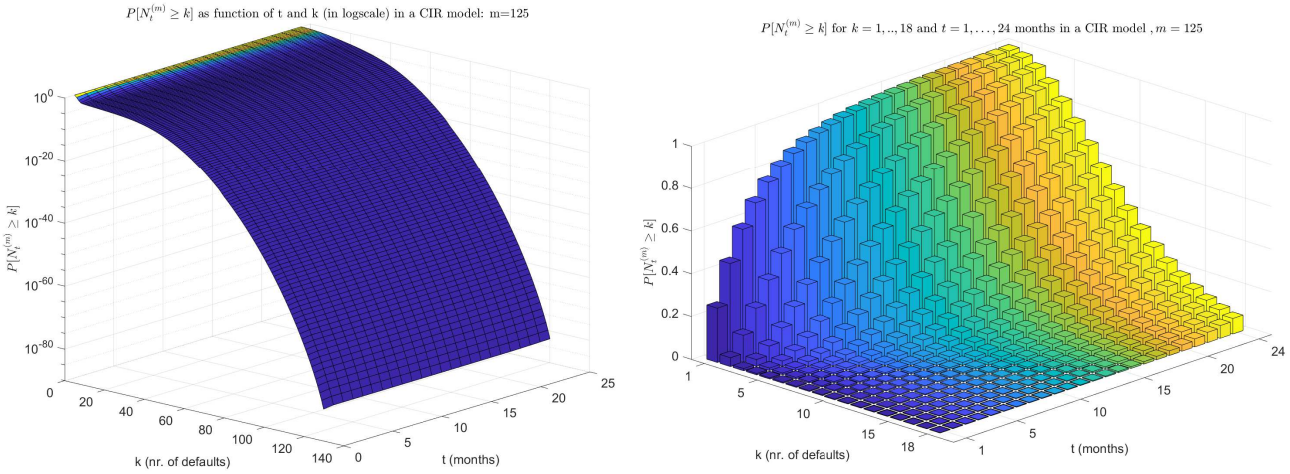


Figure 6. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ for $t = 1, 2, ..24$ months when individual default times have CIR-intensities as in Table 1 where $m = 125$. **Left panel:** in log-scale for $k = 0, ..., 125$. **Right panel:** for $k = 0, ..., 18$.

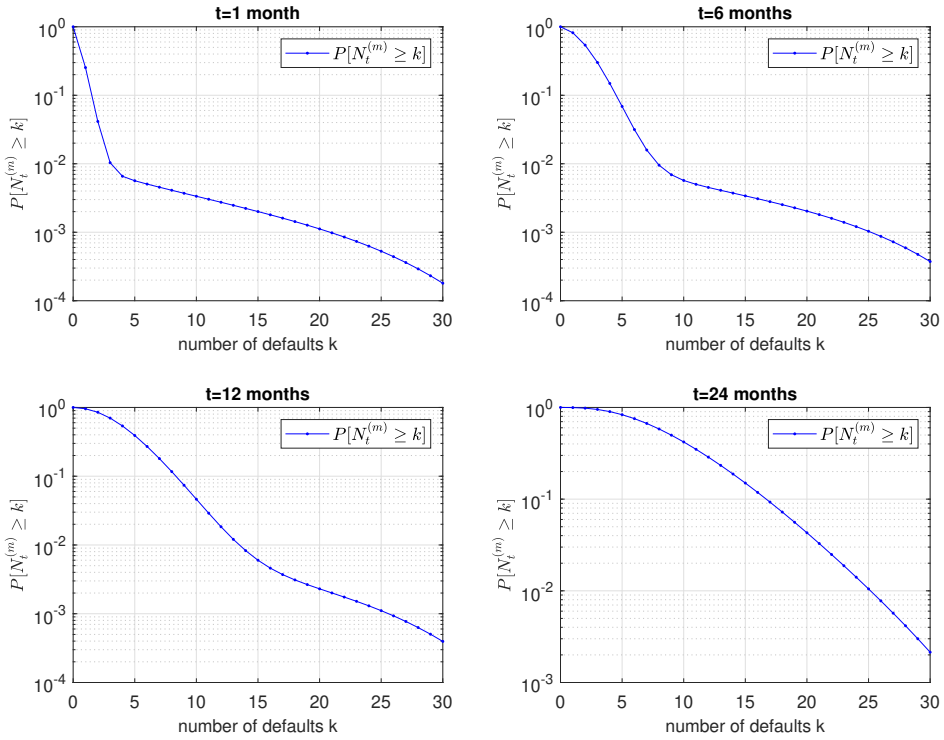


Figure 7. $m = 125$: The number of default distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in log-scale where $k = 0, ..., 30$ and $t = 1, 6, 12, 24$ months when individual default times have CIR-intensities with parameters same as in Figure 3.

As can be seen in the left panel in Figure 6, the tail probabilities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ will be quite small for large values of k, and when $k \geq 120$ then $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ is of order 10^{-80}, which are extremely small probabilities. Such small tail-probabilities follows from the choice of our CIR-parameters together with

the fact that a CIR-intensity process generally produces weak default dependence. However, the tail-probabilities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in the CIR-case are still much higher than in the case with constant default probabilities, that is, for a standard binomial distribution with independent defaults, as plotted in Figure 2 with $m = 125$ and $p = 0.0329$. To see this, we can compare the left panel in Figure 2 with the subplots in Figure 7 which displays the tail-distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in log-scale where $k = 0, ..., 30$ and $t = 1, 6, 12, 24$ months when individual default times have CIR-intensities with parameters same as in Figure 3. The tail-probabilities in all subplots of Figure 7 are up to a factor 10^{12} times bigger than the corresponding probabilities for $k = 1, \ldots, 30$ in Figure 2, where we remind that in Figure 7 the one-year default probability (i.e. for $t = 1$, or, equivalently 12 months) is 0.0329 which is same as the constant default probability p in Figure 2, that is $p = 0.0329$. In Section 9 we will see that a one-factor Gaussian copula model with one-year default probability 0.0329 (same as in the CIR-case) and moderate sizes for the correlation parameter ρ, can produce tail-probabilities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ which are up to a factor 10^{65} times bigger than the corresponding probabilities in the CIR-case.

Note that the graphs in Figure 6 and the left subplot in Figure 7 are all in log-scale. However, Figure the right panel in Figure 6 displays the time evolution of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in standard scale, where $k = 1, ..., 18$ and $t = 1, \ldots, 24$ months when individual default times have CIR-intensities with parameters same as in Figure 3.

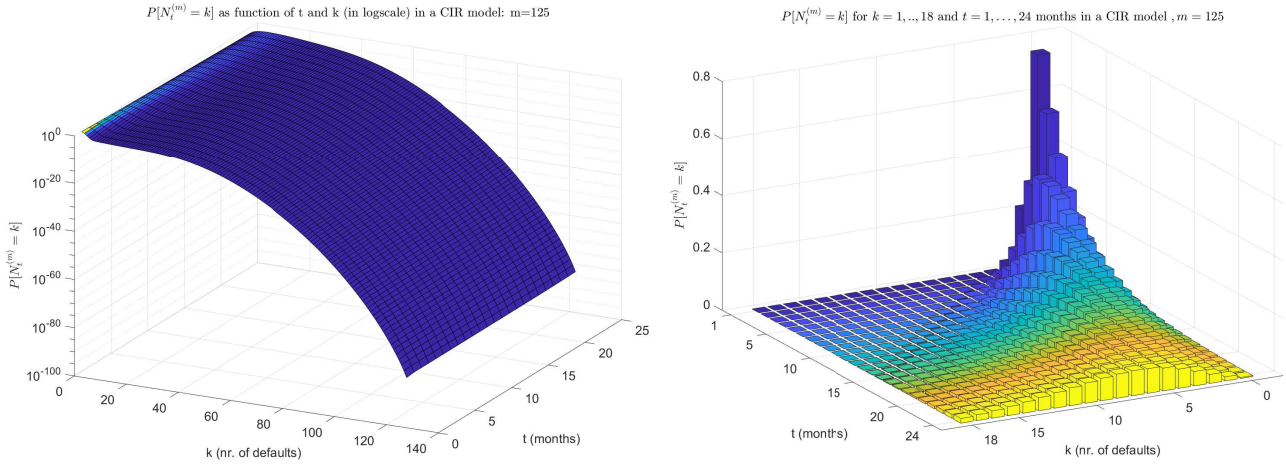


Figure 8. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $t = 1, 2, ..24$ months when individual default times have CIR-intensities as in Table 1 where $m = 125$. **Left panel:** in log-scale for $k = 0, ..., 125$. **Right panel:** for $k = 0, ..., 18$. The plots in the two panels are viewed from different angles.

Next, the left panel in Figure 8 plots, for $m = 125$, the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, \ldots, 24$ months when individual default times have CIR-intensities with parameters same as in Figure 3. Furthermore, the right panel in Figure 8 displays the time evolution of the number of distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in normal scale where $k = 1, ..., 18$ when $m = 125$ and $t = 1, \ldots, 24$ months when individual default times have CIR-intensities with parameters same as in Figure 3. All the plots in Figure 6-8 are generated via Theorem 4.1 and Corollary 4.5 and in these figures we write t in months, but the actual computations of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ are done with t measured in units of years. So for example two, six and 24 months mean that t is given by $t = \frac{2}{12}, \frac{6}{12}$ and $t = \frac{24}{12} = 2$ in our computations of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$, $\mathbb{P}\left[N_t^{(m)} = k\right]$ etc.

9. Numerical examples when the default times are driven by a one-factor Gaussian copula model

In this section we repeat the same type of computations as in Subsection 8.2, but now in a one-factor Gaussian copula model. We choose the individual one-year default probability to be same as in Subsection 8.2, which is 0.0329, that is 3.29% so that we have at least one common reference value with the CIR-case. Hence, we chose a model where the default times have conditional default distribution $p(t, Z)$ as given by (5.1.1) or alternatively, (10.1.1) where Z is a standard normal random variable. Furthermore, the unconditional default distribution $F(t)$, used in (5.1.1)-(10.1.1), is set to $F(t) = 1-e^{-\lambda t}$ with $F(1) = 0.0329$ implying that λ is given by $\lambda = 0.03345$, see also in Table 2. As for the correlation parameter ρ, we set a somewhat higher value of $\rho = 0.3$ in order to better illustrate how it is possible to create much higher probabilities compared to the CIR-case in Subsection 8.2.

Table 2. The parameters and related quantities for the one-factor Gaussian copula model and the stock prices $S_{t,j}$ where we let $m = 125$.

Gauss copula	$m = 125$	$\rho = 0.3$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$	$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$

Once we are equipped with all the parameters describing the dynamics for the one-factor Gaussian copula model, we proceed with computing the quantities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ and $\mathbb{P}\left[N_t^{(m)} = k\right]$ via Corollary 5.1.

First, for $m = 125$, the left panel in Figure 9 displays the time evolution of the tail distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, \ldots, 24$ months when individual default times are constructed via a one-factor Gaussian copula model where $m = 125$, $\rho = 0.3$ and $F(t) = 1 - e^{-\lambda t}$ and $F(1) = 0.0329$. As can be seen in the left panel of Figure 9, the tail probabilities $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ will range from 10^0 to 10^{-15}. In particular, when $k \geq 120$ then $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ will be of order 10^{-15}, which is a factor 10^{65} bigger than in the CIR-case presented in the left subplot of Figure 7, with the same individual one-year default probability of 0.0329. Furthermore, the right panel in Figure 9 displays the time evolution of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ in standard scale, where $k = 1, ..., 18$ and $t = 1, \ldots, 24$ months when individual default times have same parameters same as in the left subplot of Figure 9. The computations in Figure 9 are done via the saddlepoint approach given in Corollary 5.1.

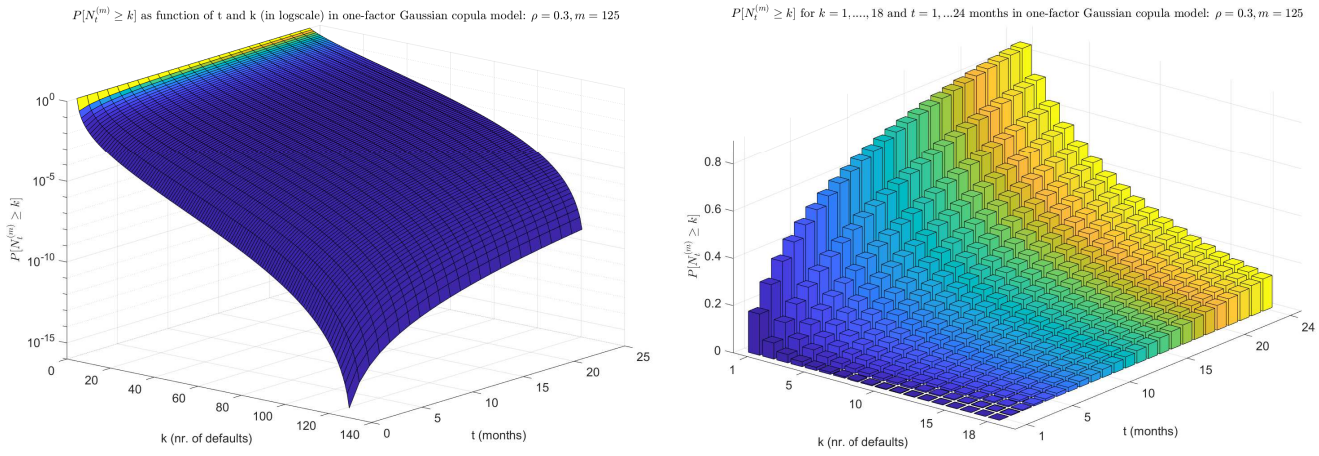


Figure 9. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ for $t = 1, 2, \ldots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 2 where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, ..., 125$. **Right panel:** for $k = 1, ..., 18$.

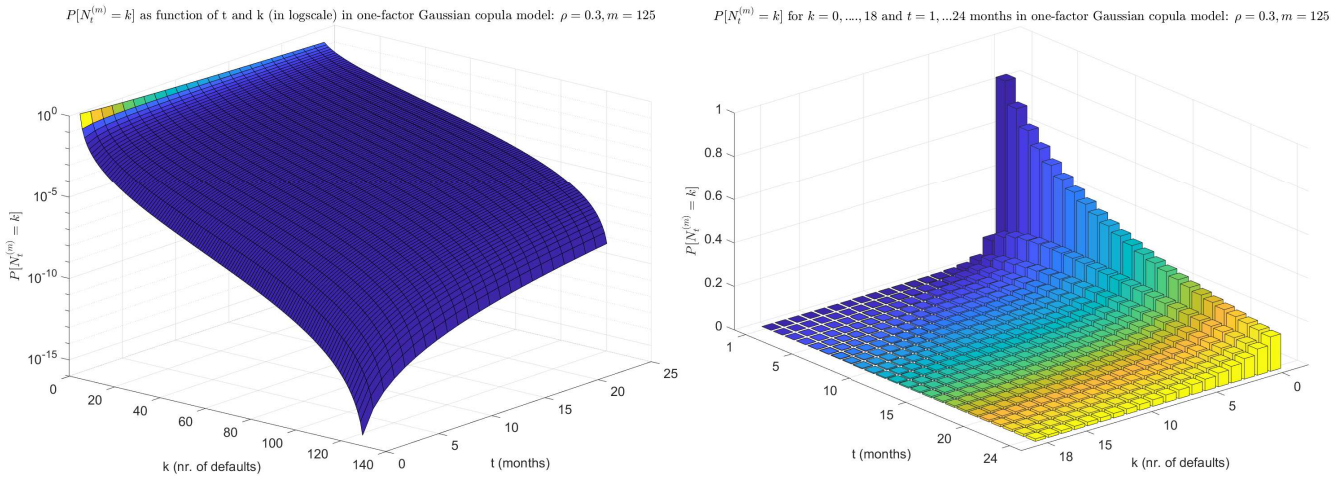


Figure 10. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $t = 1, 2, \ldots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 2 where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, ..., 125$. **Right panel:** for $k = 0, ..., 18$. The plots in the panels are viewed from different angles.

Next, the left subplot in Figure 10 displays, for $m = 125$, the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, \ldots, 24$ months when individual default times have same parameters same as in Table 2 where $m = 125$ and $\rho = 0.3$. Furthermore, the right panel in Figure 10 displays the time evolution of the number of distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in normal scale where $k = 1, ..., 18$ when $m = 125$ and $t = 1, \ldots, 24$ months when individual default times have parameters same as in the left panel of Figure 10. Note again that for larger values of k, then $\mathbb{P}\left[N_t^{(m)} = k\right]$ will be up to a factor 10^{65} bigger than the corresponding probabilites in the CIR-case presented in Figure 8, with the same individual one-year default probability of 0.0329.

All the plots in Figure 9-10 were generated via Corollary 5.1 and in these figures we write t in months, but the actual computations of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$ are done with t measured in units of years. So for example two, six and 24 months mean then t is given by $t = \frac{2}{12}, \frac{6}{12}$ and $t = \frac{24}{12} = 2$ in our computations of $\mathbb{P}\left[N_t^{(m)} \geq k\right]$, $\mathbb{P}\left[N_t^{(m)} = k\right]$ etc.

10. Numerical studies of Value-at-Risk for a large homogeneous stock portfolio where jumps in stocks are due to exogenous default times driven by a one-factor Gaussian copula model

In this section we will perform some complementary numerical studies of the stock price model developed in Herbertsson (2023*a*) which was briefly discussed in Section 7. Herbertsson (2023*a*) performs several numerical Value-at-Risk (VaR) studies which mainly focus on the time evolution of VaR for different equity portfolios holding the parameters fixed in the different credit risk models for the external group of defaultable entites negatively affecting the stock prices. In this section we will do the reverse, that is, for some few fixed time points we will study VaR in the equity portfolios as function of the "default correlation"-parameter ρ in the one-factor Gaussian copula model where ρ will continuously range in the interval $[0.02, 0.9]$. Note that the term "default correlation"-parameter ρ will sometimes, more correctly, be denoted as the Gauss copula correlation parameter ρ. However, it is easy to derive a semi-explicit expression of the true default correlation Corr $\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right)$ which will be an explicit function of Gauss copula correlation parameter ρ and the marginal default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$ which motivates why we sometimes write "default correlation"-parameter ρ instead of the term "Gauss copula correlation" parameter ρ.

Hence, in this section we will amongst others study Value-at-Risk for a portfolio of stocks for with jumps in all stock prices occurring at default times of an external group of defaultable entitles $\mathbf{C}_1, \ldots, \mathbf{C}_m$. Throughout this section we assume that the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditional independent and are driven by a one-factor Gaussian copula model with "default correlation"-parameter ρ and marginal default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$.

10.1. **The parameters and related quantities.** In this section we assume that the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditional independent and are driven by a one-factor copula model as discussed in Subsection 3.2.1. Hence, the conditional default probability is same for all entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ and given by (5.1.1), that is

$$\mathbb{P}[\tau_i \leq t \,|\, Z] = \Phi\left(\frac{\Phi^{-1}(F(t)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{10.1.1}$$

where Z is standard normal random variable, ρ is the so-called default-correlation parameter, $\Phi(x)$ is distribution function to a standard normal random variable. Furthermore, $F(t) = \mathbb{P}[\tau_i \leq t]$ is the marginal default distribution same for all entities due to the exchangeability.

The jumps $\tilde{V}_1, \ldots, \tilde{V}_m$ in the stock prices $S_{t,j}$ at the defaults $\tau_1, \tau_2 \ldots, \tau_m$ are distributed as in the numerical studies in Section 8 and Section 9, that is $\tilde{V}_i = e^{-\tilde{U}_i} - 1$ where $\tilde{U}_1, \ldots, \tilde{U}_m$ are i.i.d and exponentially distributed with parameter $\eta > 0$. Hence, given the above assumptions, the dynamics of the stock price $S_{t,j}$ is same as in Equation (7.1.1) in Definition 7.1 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $\tau_1, \tau_2 \ldots, \tau_m$ are exchangeable, conditionally independent, and come from a one-factor Gaussian model as in (10.1.1). In our numerical examples we set $F(t) = \mathbb{P}[\tau_i \leq t] = 1 - e^{-\lambda t}$ and calibrate λ so that the one-year default probability is same as in the CIR-model in Section 8, that is $0.0329 = 3.29\%$ and this gives $\lambda = 0.0335$, see in Table 3. Furthermore, the "default-correlation" ρ will vary in the interval $\rho \in [0.02,\ 0.9]$ and we let the number of defaultable entities be $m = 125$, see in Table 3.

Next we turn to the parameters for the stock price model. First, note that since we use a homogeneous stock portfolio then condition (7.2.1) is satisfies, that is the stock prices $S_{t,1}, \ldots, S_{t,J}$ satisfy $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$ and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$ in the stock portfolio. Furthermore, we let the parameters μ and σ be given by $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$ and we let the stock correlation parameter ρ_S be $\rho_S = 0.25$, see in Table 3. The jump parameter η is calibrated so that for $T = 1$ year we have

$$\mathbb{E}[S_{T,j}] = S_0 \quad \text{or equivalently} \quad \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_T^{(m)}}\right] = e^{-\mu T} \quad \text{for } T = 1 \tag{10.1.2}$$

see also Equation (7.1.6) which together with $\mathbb{E}[S_{T,j}] = S_0$ implies that $\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_T^{(m)}}\right] = e^{-\mu T}$. Hence, (10.1.2) implies that for each Gauss copula correlation ρ, the jump-parameter η is calibrated so that the defaults from the one-factor Gaussian copula model with "default correlation" ρ, "wipes" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. The calibration condition (10.1.2) implies that the jump-parameter η will be a function of the parameters μ, σ, ρ_S, S_0 and the default correlation parameter ρ. However, in this paper μ, σ, ρ_S, S_0 will all be fixed and given by Table 3, and only the default correlation parameter ρ will vary so that $\eta = \eta(\rho)$. With μ, σ, ρ_S, S_0 and λ fixed as in Table 3 with $m = 125$, and where the one-factor Gaussian copula parameter ranges in $\rho \in [0.02,\ 0.9]$ then condition (10.1.2) implies that $\eta = \eta(\rho)$ will range in the interval $\eta \in [0.1651,\ 26.22]$ and the expected value $\mathbb{E}[U_{n,j}]$ of the jumps $U_{n,j}$ in the log of stock prices as function of Gauss copula correlation ρ will range in the interval $\mathbb{E}[U_{n,j}] = \frac{1}{\eta} \in [0.03813,\ 6.058]$, see also in Figure 16 which displays $\eta = \eta(\rho)$ and $\mathbb{E}[U_{n,j}] = \frac{1}{\eta(\rho)}$ as function of ρ. The middle panel in Figure 16 displays $\mathbb{E}[U_{n,j}] = \frac{1}{\eta(\rho)}$ for $\rho \in [0.02,\ 0.9]$ and and the right panel in Figure 16 displays $\mathbb{E}[U_{n,j}]$ for $\rho \in [0.02,\ 0.8]$.

Table 3. The parameters and related quantities for the one-factor Gaussian copula model and the stock prices $S_{t,j}$ where we let $m = 125$.

Gauss copula	$\rho \in [0.02,\ 0.9]$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$	$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$	
$S_{t,j}$	$S_{0,j} = 50$	$\mu = 0.15$	$\sigma = 0.2$	$\rho_S = 0.25$	$\eta \in [0.1651,\ 26.22]$	$\mathbb{E}[U_{n,j}] = \frac{1}{\eta}$

10.2. **VaR for fixed time periods for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a one-factor Gaussian copula model.** In this subsection we study Value-at-Risk (VaR) for an equally value weighted homogeneous portfolio of $J = 150$ stocks issued by the J firms $\mathbf{A}_1, \dots, \mathbf{A}_J$ with stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$ that have jumps occurring at default times $\tau_1, \tau_2 \dots, \tau_m$ which are exchangeable, conditional independent and are driven by a one-factor Gaussian copula model as discussed in Subsection 10.1 and with parameters as in Table 3. In particular, we study VaR for the above stock portfolio as function of the one-factor Gaussian copula parameter ρ for the external group of defaultable entitles $\mathbf{C}_1, \dots, \mathbf{C}_m$ with default times $\tau_1, \tau_2 \dots, \tau_m$ which negatively affects the stock prices $S_{t,1}, S_{t,2}, \dots, S_{t,J}$. For each default correlation ρ, the distribution of the jumps in the stock prices will have the same jump-parameter η determined by the condition (10.1.2). We here remark that it is of course also possible to fix the jump-parameter η to be the same value for all default correlation parameters ρ. However, this will imply different expected values $\mathbb{E}[S_{T,j}]$ of the stock prices for different ρ-values and it will then be difficult to realistically compare VaR-values for the stock portfolio losses across different default correlation parameters ρ. In this paper we choose to make the stock portfolio VaR-losses to be done under somewhat economically sound and interpretable conditions. We believe that the condition (10.1.2), that is $\mathbb{E}[S_{T,j}] = S_0$ where $T = 1$ year, for all default correlation parameters ρ, is an economically equal and interpretable condition which makes it intuitive clear that the expected value of the loss will stay the same at $T = 1$ year, thus implying that the jump parameter η must change as the default correlation parameters ρ increases. Of course, condition (10.1.2) can be replaced with another condition, for example having the same constant jump-parameter η, but this will then imply different values of $\mathbb{E}[S_{T,j}]$ for different default correlation parameters ρ.

Since our equity portfolio is large and homogeneous, we will in our stock portfolio VaR computations use the LPA approximation formulas in Theorem 7.8. Figures 11 - 12 displays Value-at-Risk in % of V_0 as function of the one-factor Gaussian copula parameter $\rho \in [0.02, 0.55]$ four different time points $t = 5, 10, 20, 40$ days and the three different confidence levels $\alpha = 95\%, 99\%$ and $\alpha = 99.9\%$. In Figures 11 - 12 we put an upper restriction of ρ at 0.55 but in Figure 15 this is relaxed to so that $\rho \in [0.02, 0.90]$. In the end of this section we will discuss in more detail why $\rho \leq 0.55$ in Figures 11 - 12.

The left panel in Figure 11 displays Value-at-Risk (in % of V_0) for $\alpha = 95\%, 99\%$ and $\alpha = 99.9\%$ at $t = 5$ days as function of Gauss copula correlation ρ where VaR is computed with the LPA-formula in Theorem 7.8 for a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3. For each ρ condition (10.1.2) holds, that is $E[S_{T,j}] = S_0$ for $T = 1$ year. The right panel in Figure 11 displays same quantities as in left panel, but now for $t = 10$ days. Similarly, the two panels in Figure 12 displays the same type of VaR values as in Figures 11 but now for $t = 20$ days (left panel in Figure 12) and $t = 40$ days (right panel in Figure 12).

The interpretation of all the curves in Figure 11 - 12 are done in the same way. For example, in the left panel of Figure 11, looking at the red line (99.9%-VaR) and when the one-factor Gaussian correlation is $\rho = 0.5 = 50\%$, we see that after 5 days, then there is a 0.1% probability of having a loss in the stock portfolio which is 44.77% or bigger, of the initial portfolio value V_0 at time $t = 0$. Similarly, from the black-dotted line (99%-VaR) and when the one-factor Gaussian correlation is $\rho = 0.5 = 50\%$, we see that after 5 days, then there is a 1% probability of having a loss in the stock portfolio which is 10.69% or bigger, of the initial portfolio value V_0 at time $t = 0$. Furthermore, moving to VaR for $t = 40$ days (i.e. two trading months) displayed in the right panel of Figure 11 with $\rho = 0.5 = 50\%$, looking at the red line (99.9%-VaR) we see that after 40 days, then there is a 0.1% probability of having a loss in the stock portfolio which is 90% or bigger, of the initial portfolio value V_0 at time $t = 0$.

In order to benchmark our VaR-results in Figures 11 - 12 we also compare our results with the corresponding VaR-values in the Black-Scholes case with same drift and volatilises as in the stock price model with jump at defaults. Thus, Table 4 displays Value-at-Risk in % of V_0 at four different time points $t = 5, 10, 20, 40$ days and the three different confidence levels $\alpha = 95\%, 99\%$ and $\alpha = 99.9\%$, computed with the Black-Scholes LPA-formula in Equation (7.2.12) with parameters as in Table 3. As can be seen in Table 4 and comparing with the VaR-levels in Figure 11 - 12, the differences between the jump vs non-jump VaR-cases are huge, sometimes several hundred times bigger for larger ρ-values.

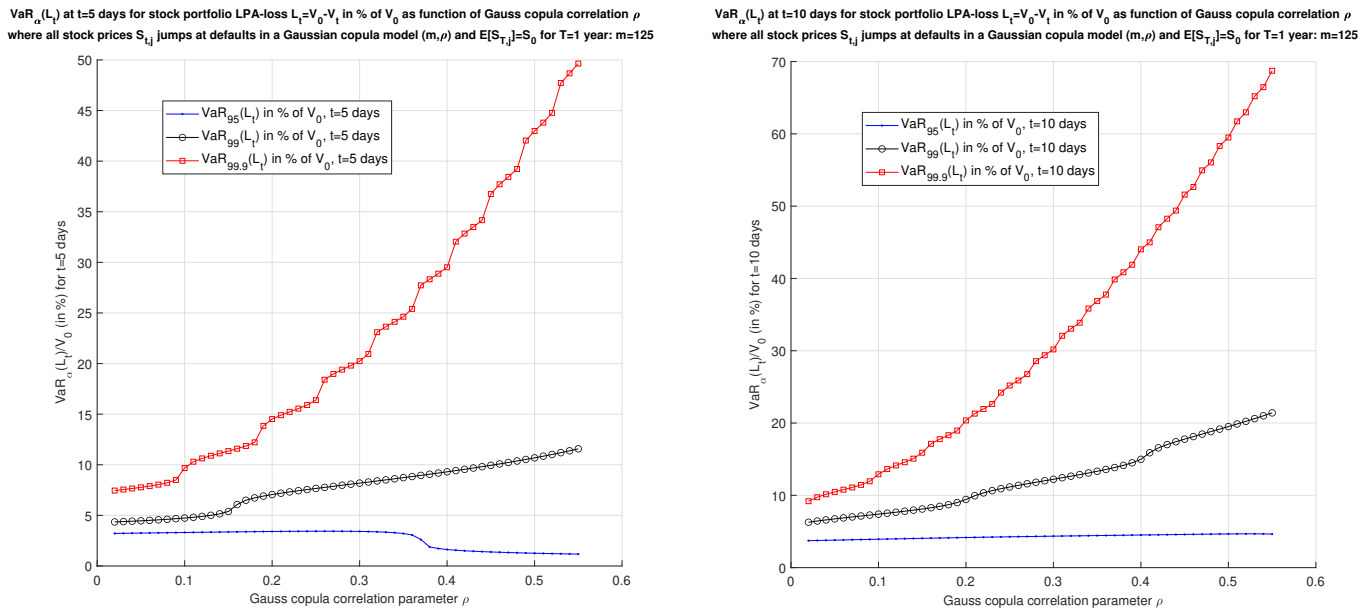


Figure 11. Value-at-Risk (in % of V_0) at fixed time point $t = 5$ and $t = 10$ days as function of Gauss copula correlation ρ where VaR is computed with the LPA-formula in Theorem 7.8 for a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3. For each ρ condition (10.1.2) holds, that is $E[S_{T,j}] = S_0$ for $T = 1$ year. **Left panel:** VaR for $t = 5$ days. **Right panel:** VaR for $t = 10$ days.

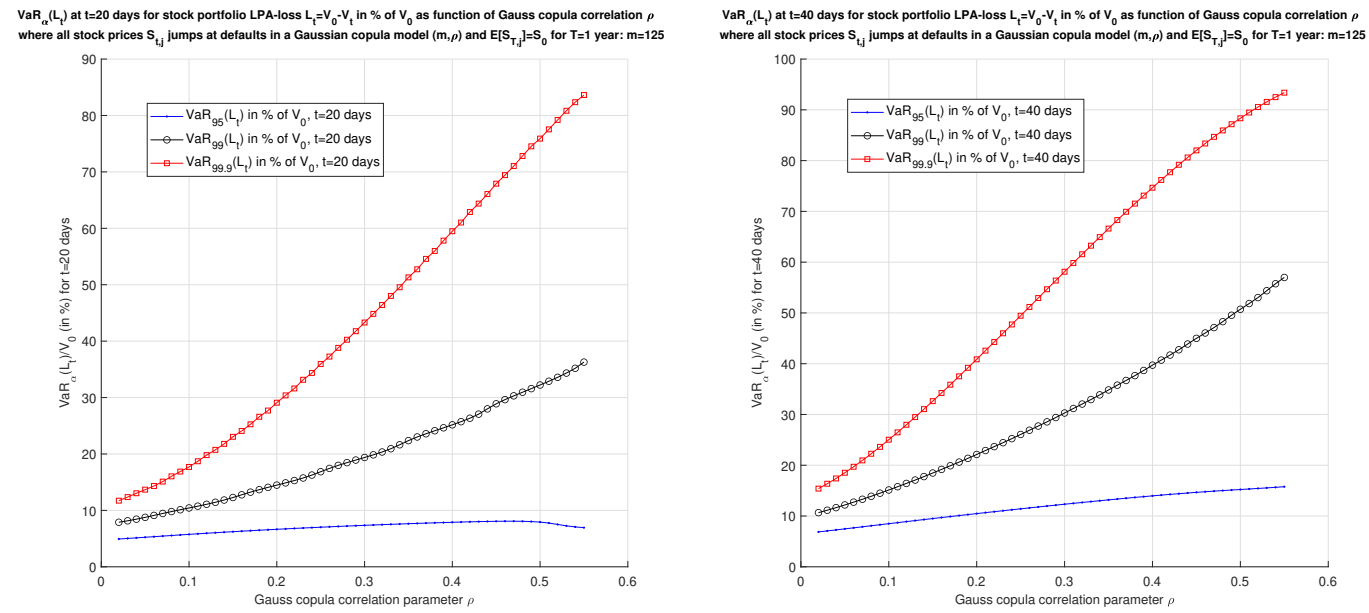


Figure 12. Value-at-Risk (in % of V_0) at fixed time point $t = 20$ and $t = 40$ days as function of Gauss copula correlation ρ where VaR is computed with the LPA-formula in Theorem 7.8 for a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3. For each ρ condition (10.1.2) holds, that is $E[S_{T,j}] = S_0$ for $T = 1$ year. **Left panel:** VaR for $t = 20$ days. **Right panel:** VaR for $t = 40$ days.

For example, in Table 4 we see that at $t = 10$ days the Black-Scholes 99.9%-VaR LPA-formula is 2.457% of the initial stock portfolio value V_0 at time $t = 0$ while in the stock price model with jumps, at $t = 10$ days and with default correlation $\rho = 0.55$ (see right panel in Figure 11), then the 99.9%-VaR

LPA-formula (computed via Theorem 7.8) is 68.73% of the initial stock portfolio value V_0 at time $t = 0$. Thus, the relative difference between the Black-Scholes VaR and VaR with jumps is then over 2697% if we measure the relative difference with respect to the Black-Scholes case ($100 \times \frac{68.73-2.457}{2.457} = 2697$). We also remind that a negative loss is a gain implying that the Black-Scholes case we see that in the Black-Scholes LPA portfolio model it is extremely difficult to obtain losses over a two-month period (40 days), as seen in Table 4.

Table 4. Value-at-Risk (in % of V_0) for $t = 5.10, 20, 40$ days of a homogeneous portfolio with $J = 150$ stocks in the Black-Scholes case computed with the LPA-formula in Equation (7.2.12) and with drift, volatility and stock correlation as in Table 3.

t (in days)	5	10	20	40
Black-Scholes $\text{VaR}_{95\%}\left(L_t^{(V)}\right)$ (in % of V_0)	0.8596	1.0426	1.1299	-6.3331
Black-Scholes $\text{VaR}_{99\%}\left(L_t^{(V)}\right)$ (in % of V_0)	1.3343	1.7120	2.0745	-2.8553
Black-Scholes $\text{VaR}_{99.9\%}\left(L_t^{(V)}\right)$ (in % of V_0)	1.8637	2.4570	3.1225	0.9080

In the left panels of Figures 11 - 12 we see that the 95%-VaR is not monotonically increasing in the "default correlation"-parameter ρ. The non-monotonic VaR in ρ is even more clear in the left (first) panel of Figure 13.

To investigate this further, we plot in the left panel of Figure 14, the stock-portfolio 95%-VaR (in % of V_0) as function of Gauss copula correlation $\rho \in [0.02, 0.9]$ at the time points $t = 10, 20, 40$ days and where VaR is computed with the LPA-formula in Theorem 7.8 for a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3.

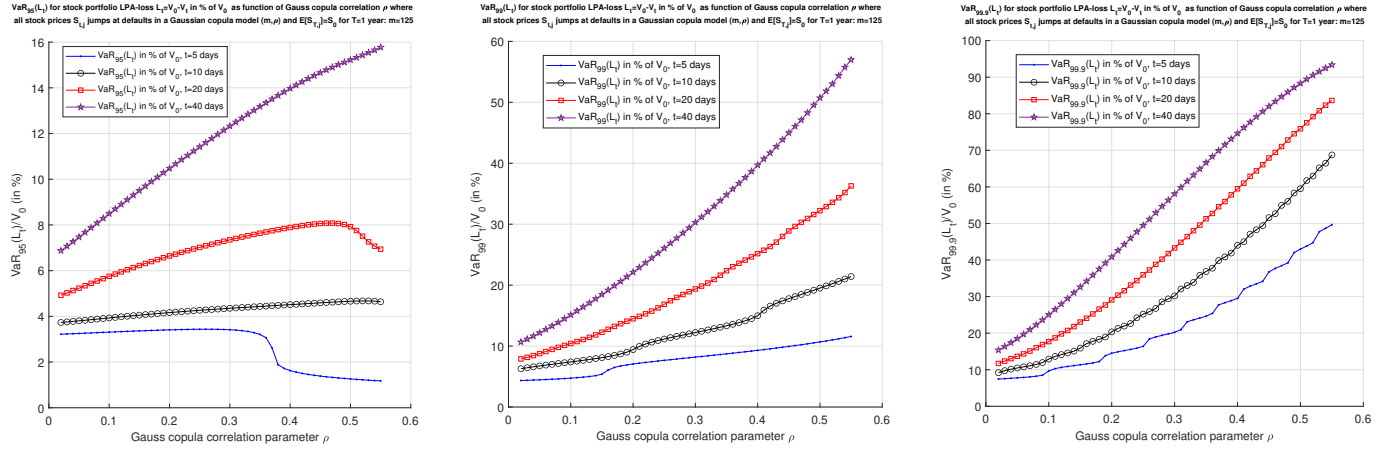


Figure 13. Value-at-Risk (in % of V_0) at fixed confidence level $\alpha = 95\%, 99\%$ and $\alpha = 99.9\%$ as function of Gauss copula correlation ρ for $t = 5, 10, 20, 40$ days where VaR is computed with the LPA-formula in Theorem 7.8 for a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3. For each ρ condition (10.1.2) holds, that is $E[S_{T,j}] = S_0$ for $T = 1$ year. **Left panel:** VaR for $\alpha = 95\%$. **Middle panel:** VaR for $\alpha = 99\%$. **Right panel:** VaR for $\alpha = 99.9\%$.

From left panel of Figure 14 it is clear that the stock-portfolio 95%-VaR is not only non-monotonic, it can also has multiply local maxima depending on the time point t. The are two main reason for the fluctuating stock-portfolio 95%-VaR. The first reason is that the LPA stock portfolio loss distribution $F^{\text{LPA}}_{L_t^{(V)}}(x)$ defined in Equation (7.2.7), is a weighted sum consisting of $m + 1$ terms where for each term, indexed by $k = 0, 1, \ldots, m$, is a product of $\mathbb{P}\left[N_t^{(m)} = k\right]$ and a probability $\Phi_k(x, t, \eta, \ldots) = 1 - \Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right) - \left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)$, see also in (7.2.7). Now, in the one-factor Gaussian copula model

and under the calibration condition 10.1.2, both sequences $\left(\mathbb{P}\left[N_t^{(m)}=k\right]\right)_{k=0}^m$ and $(\Phi_k(x,t,\eta,\ldots))_{k=0}^m$ will be functions of the default correlation parameter ρ since $\eta=\eta(\rho)$ in view of 10.1.2. As ρ increases the weights $\left(\mathbb{P}\left[N_t^{(m)}=k\right]\right)_{k=0}^m$ will change for changing ρ and in particular, will create a more and more U-shaped form as function of k for fixed t as ρ increases and the U-shape will also be more pronounced as time t increases, see e.g in Figure 18 and Figure 19. These observations for the "weights" $\mathbb{P}\left[N_t^{(m)}=k\right]$ will then imply that the for a fixed x, the mapping $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)=F^{\mathrm{LPA}}_{L_t^{(V)}}(x;\rho)$ viewed as a mapping of ρ, will not be monotonic in ρ and consequently, also the inverse $(F^{-1})^{\mathrm{LPA}}_{L_t^{(V)}}(\alpha;\rho)$ for fixed α and fixed t, seen as a function of ρ, will not necessarily be monotonic in ρ, as is clearly seen in the left panel of Figure 14. The second reason for the fluctuating stock-portfolio 95%-VaR and in particular the decreasing 95%-VaR for higher ρ-values, in particular as ρ approaches 90% is due to the fact that the 95%-VaR for the number of defaults $N_t^{(m)}$ is a also non-monotonic in ρ, and the 95%-VaR for $N_t^{(m)}$ will be zero for higher ρ-values when $t=10,20,40$ days, as is clearly seen in the right panel of Figure 14. Furthermore, also the 99%-VaR for the number of defaults $N_t^{(m)}$ is non-monotonic in ρ as seen in the left subplot of Figure 15. However, for the 99.9%-VaR for $N_t^{(m)}$ will be strictly increasing in ρ when $t=10,20,40$ days, which is seen in the right panel of Figure 15.

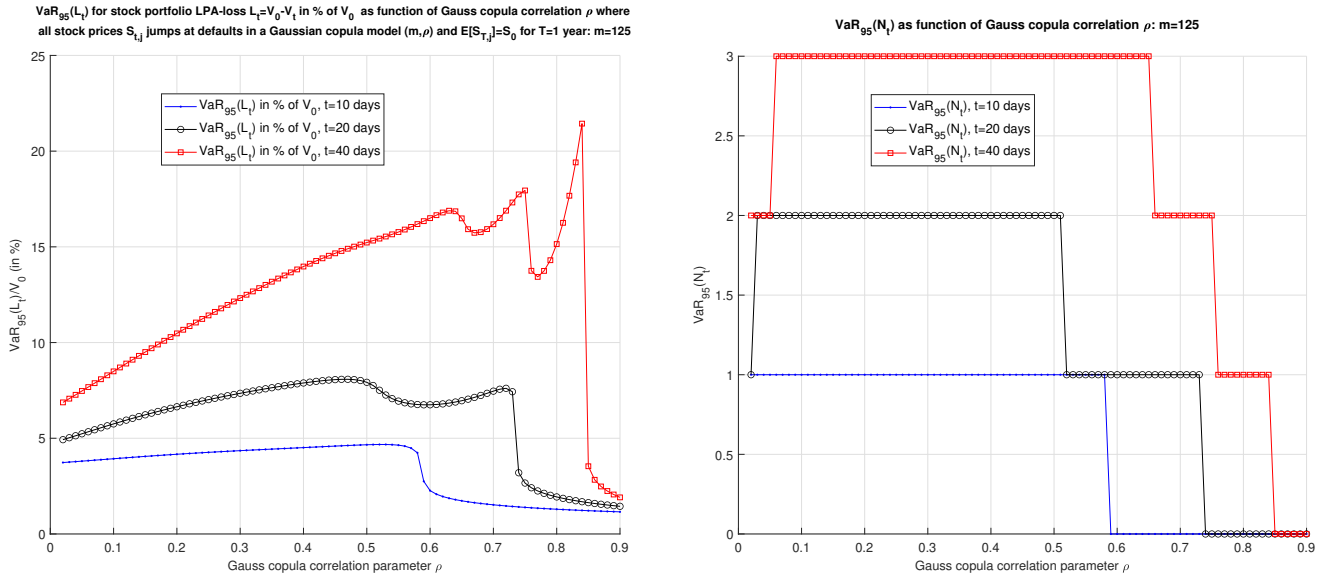


Figure 14. Left panel: Value-at-Risk (in % of V_0) at $\alpha=95\%$ as function of Gauss copula correlation ρ for $t=10,20,40$ days where VaR is 7.8 for a homogeneous portfolio with $J=150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m=125$ and parameters as in Table 3. For each ρ condition (10.1.2) holds, that is $E[S_{T,j}]=S_0$ for $T=1$ year. **Right panel:** Value-at-Risk at $\alpha=95\%$ for $N_t^{(m)}$ as function of Gauss copula correlation ρ for $t=5,10,20$ and with parameters as in Table 3 where $m=125$.

The motivation for zero 95%-VaR for the number of defaults $N_t^{(m)}$ at very high ρ-values for $t=10,20,40$, as seen in the right panel of Figure 14, is due to the fact that individual default probabilities at these time points are small, and the very large ρ-value will make $\mathbb{P}\left[N_t^{(m)}=0\right]>95\%$. For example, at $t=10$ days and $\rho=0.8$ then $\mathbb{P}\left[N_t^{(m)}=0\right]=97.82\%$, and at $t=20$ days and $\rho=0.8$ then $\mathbb{P}\left[N_t^{(m)}=0\right]=96.26\%$ and at $t=40$ days and $\rho=0.87$ then $\mathbb{P}\left[N_t^{(m)}=0\right]=95.69\%$ and all these observations implies that 95%-VaR will be zero at these time points and ρ-values.

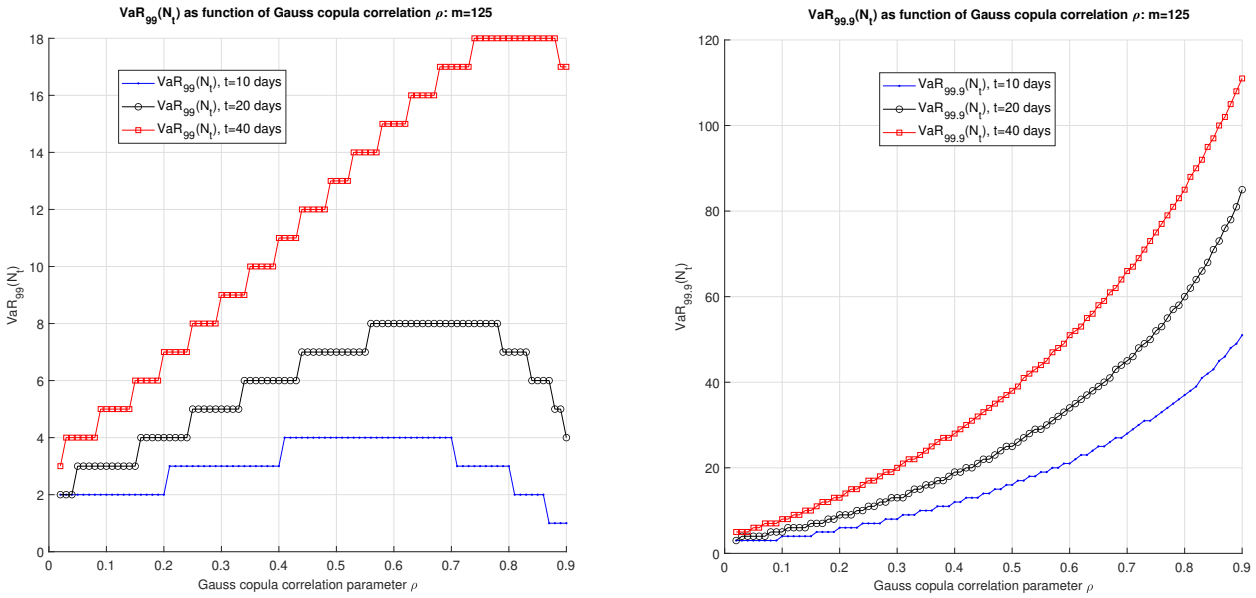


Figure 15. Value-at-Risk for $N_t^{(m)}$ at fixed confidence level α as function of Gauss copula correlation ρ for $t = 10, 20, 40$ days and with parameters as in Table 2 where $m = 125$. **Left panel:** VaR for $\alpha = 99\%$. **Right panel:** VaR for $\alpha = 99.9\%$.

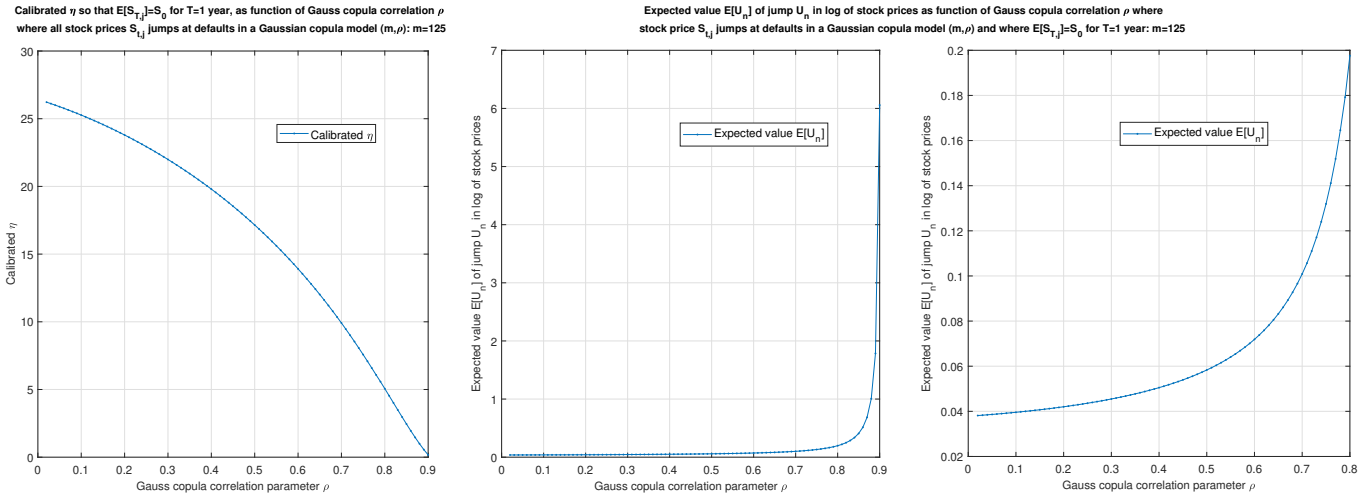


Figure 16. **Left panel:** Calibrated η so that $E[S_{T,j}] = S_0$ for $T = 1$ year as function of Gauss copula correlation ρ where the stock price $S_{t,j}$ jumps at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3. **Middle panel:** Expected value $E[U_n] = \frac{1}{\eta}$ of jumps U_n in log of stock prices as function of Gauss copula correlation ρ where the stock price $S_{t,j}$ jumps at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 3 and where η for each ρ is given by the left panel. **Right panel:** Same as middle panel but where ρ runs up to $\rho = 0.8$.

The reason for letting $\rho \leq 0.55$ in Figures 11 - 12 is that VaR-equation for $\alpha = 99\%$ and 99.9% will already for ρ around 0.60 reach the maximum loss of V_0, which due to the LPA-VaR formula (7.2.9) will in the solution of the nonlinear equation $(F^{-1})^{\text{LPA}}_{L_t^{(V)}}(\alpha)$ produce numerical instabilities since the distribution to the LPA approximation of the stock portfolio loss $F^{\text{LPA}}_{L_t^{(V)}}(x)$ will be close to its maximum valule of $V_0 = JS_0$ when ρ grows and when $\alpha = 99\%$ and 99.9%. It is possible to overcome this problem simply by manually solve the equation $F^{\text{LPA}}_{L_t^{(V)}}(x) = \alpha$ using e.g. a while loop but this is a very time consuming

procedure which also leads to a non-smooth curve of VaR for larger ρ. However, for lower confidence levels such as e.g. $\alpha = 95\%$ the numerical solution to $F^{\text{LPA}}_{L_t^{(V)}}(x) = \alpha$, that is the value $(F^{-1})^{\text{LPA}}_{L_t^{(V)}}(\alpha)$ is obtained non-problematic also for higher values of ρ, as can be seen in the left panel of Figure 14 which shows the 95%-VaR for $\rho \in [0.02, 0.90]$ in steps of 0.01 at four different time points $t = 5, 10, 20, 40$. Similarly, the right panel in Figure 14 shows the 99%-VaR for $t = 5, 10, 20, 40$ days, with same parameters as in the left and middle panels of Figure 14.

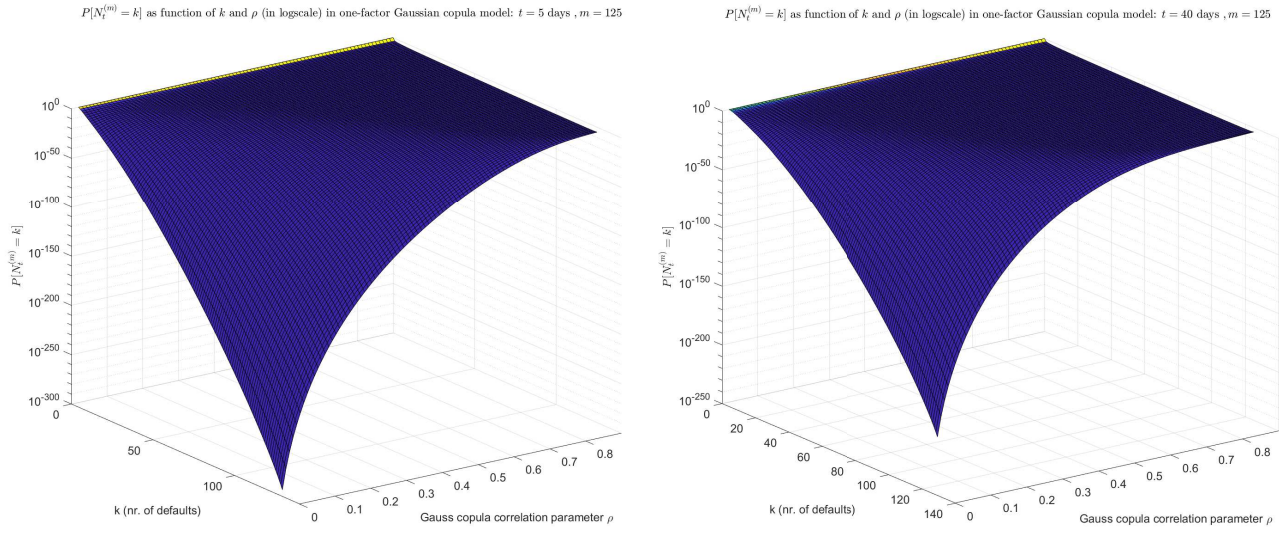


Figure 17. The distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale for $k = 0, ..., 125$ as function of Gauss copula correlation $\rho \in [0.02, 0.9]$ with parameters as in Table 3 where $m = 125$. **Left panel:** For $t = 5$ days. **Right panel:** For $t = 40$ days.

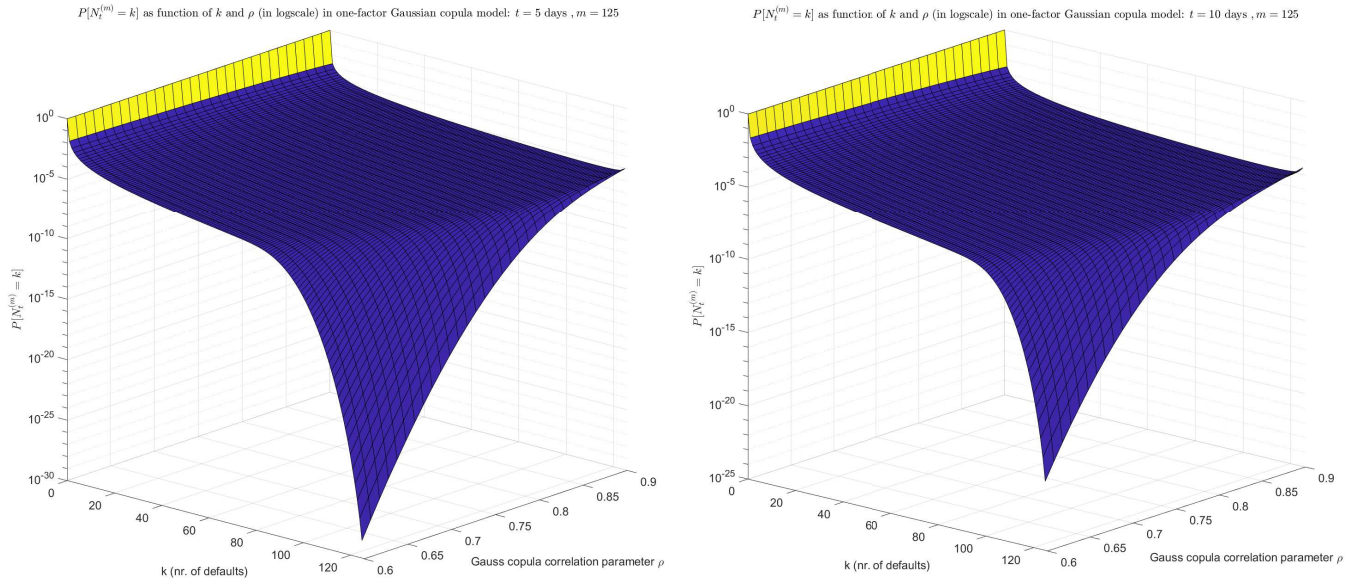


Figure 18. The distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale for $k = 0, ..., 125$ as function of Gauss copula correlation $\rho \in [0.6, 0.9]$ with parameters as in Table 3 where $m = 125$. **Left panel:** For $t = 5$ days. **Right panel:** For $t = 10$ days.

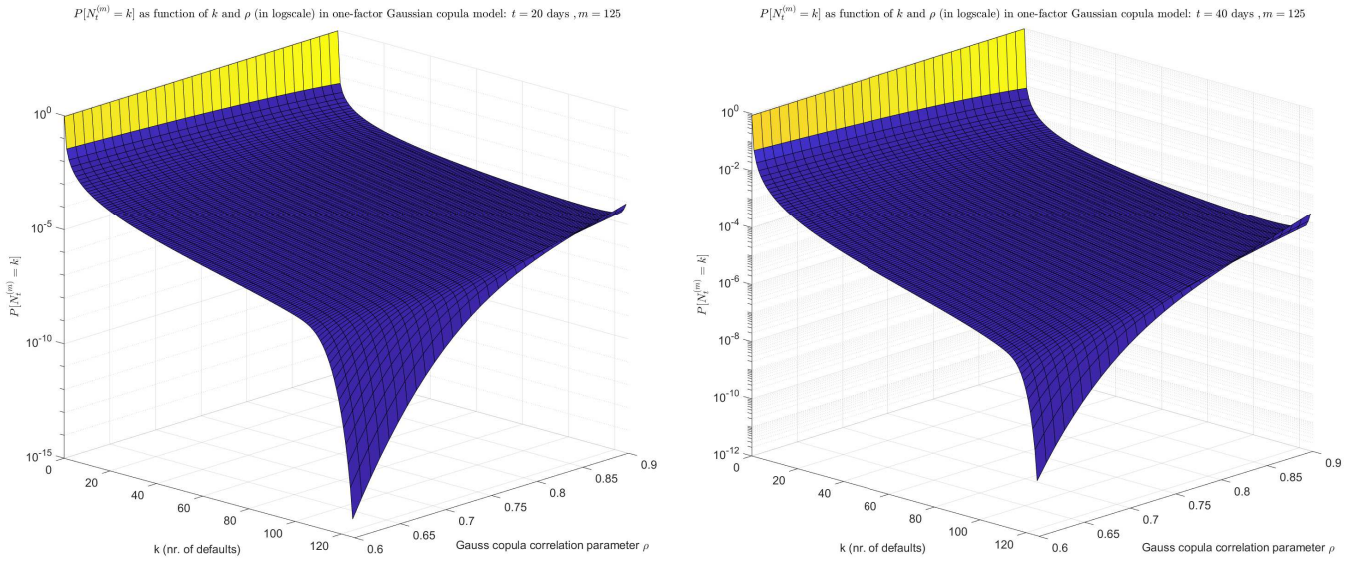


Figure 19. The distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale for $k = 0, ..., 125$ as function of Gauss copula correlation $\rho \in [0.6, 0.9]$ with parameters as in Table 3 where $m = 125$. **Left panel:** For $t = 20$ days. **Right panel:** For $t = 40$ days.

We finally remark that all computations done in all figures from Figure 11 to Figure 16 would be very difficult to efficiently and in rather limited time be done without having the efficient saddlepoint algorithms for computing the probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ as function of ρ and k, which are displayed in Figure 17 - Figure 19 for various time points t and different regions for the "default correlation" parameter ρ, which we sometimes also denote by the Gauss copula correlation parameter ρ.

11. Comparing the saddlepoint method with other numerical methods

In this section we compare the method of computing $\mathbb{P}\left[N_t^{(m)} = k\right]$ via the saddlepoint approach with another numerical method, briefly discussed in Subsection 3.2.1. For simplicity, we will only focus on the one-factor Gaussian copula model, mainly since the factor will be unchanged by the parameters so it is easy and quick to change the parameters and redo computations (while in e.g. the CIR-case a new Fourier inversion has to be done to find the time-dependent factor-density for each new parameter setting).

The alternative method for computing $\mathbb{P}\left[N_t^{(m)} = k\right]$ presented in this section is purely numeric and lacks the analytical formulas obtained in the saddlepoint approach. Below we present the alternative method for computing $\mathbb{P}\left[N_t^{(m)} = k\right]$. Recall from Equation (3.2.11), now adapted to a factor model with conditional default probability $p(t, Z)$, that

$$\mathbb{P}\left[N_t^{(m)} = k\right] = \mathbb{E}\left[\binom{m}{k} p(t,Z)^k \left(1 - p(t,Z)\right)^{m-k}\right] = \int_{-\infty}^{\infty} \binom{m}{k} p(t,z)^k \left(1 - p(t,z)\right)^{m-k} f_Z(z)\, dz \quad (11.1)$$

where $p(t, Z)$ for example is given by (10.1.1) or (5.2.1.2) and $f_Z(z)$ is the density to the random factor Z driving the conditional default probability $p(t, Z)$. As discussed in Subsection 3.2.1, a direct computation of the integral in the right hand side of (11.1) will for say, $m \geq 55$ in practice not work since the binomial coefficient will be to large to be stored with exact accuarcy on a standard computer, using standard math software such as e.g. `Matlab`, `R` or Python. But numerically, one can in most software packages circumvent the problem discussed in connection with Equation (11.1) by e.g. use built-in functions for the probability distribution of a binomial distribution applied to the "fixed" probability $p(t, z)$ and then repeat this for every z in the numerical quadrature of the integral in the right hand side of (11.1). For example, in matlab the quantity $\binom{m}{k} p(t,z)^k \left(1 - p(t,z)\right)^{m-k}$ can be computed via the built-in numerical functions such as e.g. `binopdf`, with the binomial parameters m and $p(t, z)$ and the integer k and then use this in a numerical approximation of the integral in (11.1). Hence, if we consider a mesh $z_1 < z_2 < \ldots z_{J_C}$

on a limited interval of the range of Z, which in our case will be the real line $\mathbb{R}$, we can then approximate the integral (11.1) as follows,

$$\begin{aligned}\int_{-\infty}^{\infty}\binom{m}{k}p(t,z)^k\left(1-p(t,z)\right)^{m-k}f_Z(z)\,dz &\approx \sum_{n=1}^{J_C}\binom{m}{k}p(t,z_n)^k\left(1-p(t,z_n)\right)^{m-k}f_Z(z_n)\Delta z_n \\ &= \sum_{n=1}^{J_C}\underbrace{\text{binopdf}(k,m,p(t,z_n))}_{=\binom{m}{k}p(t,z_n)^k(1-p(t,z_n))^{m-k}}f_Z(z_n)\Delta z_n\end{aligned} \tag{11.2}$$

which together with (11.1) implies that

$$\mathbb{P}\left[N_t^{(m)}=k\right]\approx\sum_{n=1}^{J_C}\underbrace{\text{binopdf}(k,m,p(t,z_n))}_{=\binom{m}{k}p(t,z_n)^k(1-p(t,z_n))^{m-k}}f_Z(z_n)\Delta z_n\,. \tag{11.3}$$

We will now use the approximation (11.3) as a benchmark to the saddlepoint formula for $\mathbb{P}\left[N_t^{(m)}=k\right]$ derived in this paper. Since we in this section only consider a one-factor Gaussian factor model then $f_Z(z)$ will be the density to a standard normal random variable and we will thus compare the numerical benchmark method in (11.3) with the saddlepoint Equation (5.3.6), that is

$$\mathbb{P}\left[N_t^{(m)}=k\right]\approx\int_{-\infty}^{\infty}\Delta H_B^{(\text{LR})}\left(k,m,p(t,z)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz\approx\sum_{n=1}^{J_C}\Delta H_B^{(\text{LR})}\left(k,m,p(t,z_n)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z_n^2}{2}}\Delta z_n \tag{11.4}$$

where the mappings $H_B^{(\text{LR})}(x,m,p)$ and $\Delta H_B^{(\text{LR})}(k,m,p)$ in (11.4) are defined as in Corollary 5.1. Note that the first approximation in (11.4) is obtained by dropping the error-term $O(m^{-3/2})$ in (5.3.6). We also observe that (11.4) can be obtained by applying the approximation (5.3.5) for $\mathbb{P}\left[\bar{N}_t^{(m)}\geq x\right]$ with $x=\frac{k}{m}$ and $x=\frac{k+1}{m}$ and then take the difference of these two-tail probabilities, and finally dropping the error-term $O(m^{-3/2})$ in (5.3.5). Furthermore, the mesh in (11.4) should be approximately the same as in (11.3). In practice, when computing the numerical integrals both in (11.3) and (11.4) we use the built-in matlab-function `integral` over roughly the same interval, which thus takes care of the creating of the mesh. Furthermore, since the density $f_Z(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ will have very thin tails, we don't have to consider a very large mesh on $\mathbb{R}$ for the integrals in (11.3) and (11.4) since $f_Z(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ will be extremely small outside e.g. the interval $[-20,20]$. Regarding the numerical routine `binopdf` in matlab, the help function in Matlab does not specify which type of method that is used in `binopdf`. However, in Loader (2002), the authors claims that `binopdf` is based on a log-transform method of the exact expression together with the fact that $\ln(m!)$ can be computed via the Gamma function and applying $\ln(\cdot)$ on this expression, see p.2 in Loader (2002). Regarding the case $k=m$, we remind that the saddlepoint approach in this paper will give an exact expression for $\mathbb{P}\left[N_t^{(m)}=m\,\middle|\,Z\right]$ since $\mathbb{P}\left[N_t^{(m)}=m\,\middle|\,Z\right]=p(t,Z)^m$, and this will in turn completely remove the error term $O(m^{-3/2})$, when computing $\mathbb{P}\left[N_t^{(m)}=m\right]$ via the integral in (11.4). Hence, the first approximation in (11.4) will turn into an equality when $k=m$, and since we use the same type of discretization of the integral in (11.4) and the `binopdf`-method (11.3), then, at least in theory, we should for $k=m$, have zero difference between the two methods (11.3) and (11.4).

Next, Figure 21 - 23 displays some computations of $\mathbb{P}\left[N_t^{(m)}=k\right]$ via the saddle point method in (11.4) and the `binopdf`-method in (11.3) and their relative errors, in a one-factor Gaussian copula model for different parameters.

First, Figure 20 displays the case $m=30$, $t=4$ months, i.e. $t=\frac{4}{12}$, in a one-factor Gaussian copula model where $\rho=0.3$ and $F(t)=1-e^{-\lambda t}$ with $F(1)=0.0329$. The left panel of Figure 20 displays $\mathbb{P}\left[N_t^{(m)}=k\right]$ both via the saddlepoint approach (via Equation (11.4)) and Matlabs built in method binopdf (via Equation (11.3)) where $k=0,...,m$ with $m=30$ while the left panel of Figure 20 shows the relative difference (in percent) between the two methods where the relative difference is measured with respect to the binopdf-method. As can be in the left panel of Figure 20 the errors for $k=0,...,28$ and

$k = 30$ are all under 1.89% while for $k = 29$ the error is 8.49% where $\mathbb{P}\left[N_t^{(30)} = 29\right] = 1.2336 \times 10^{-9}$ in the binopdf-case.

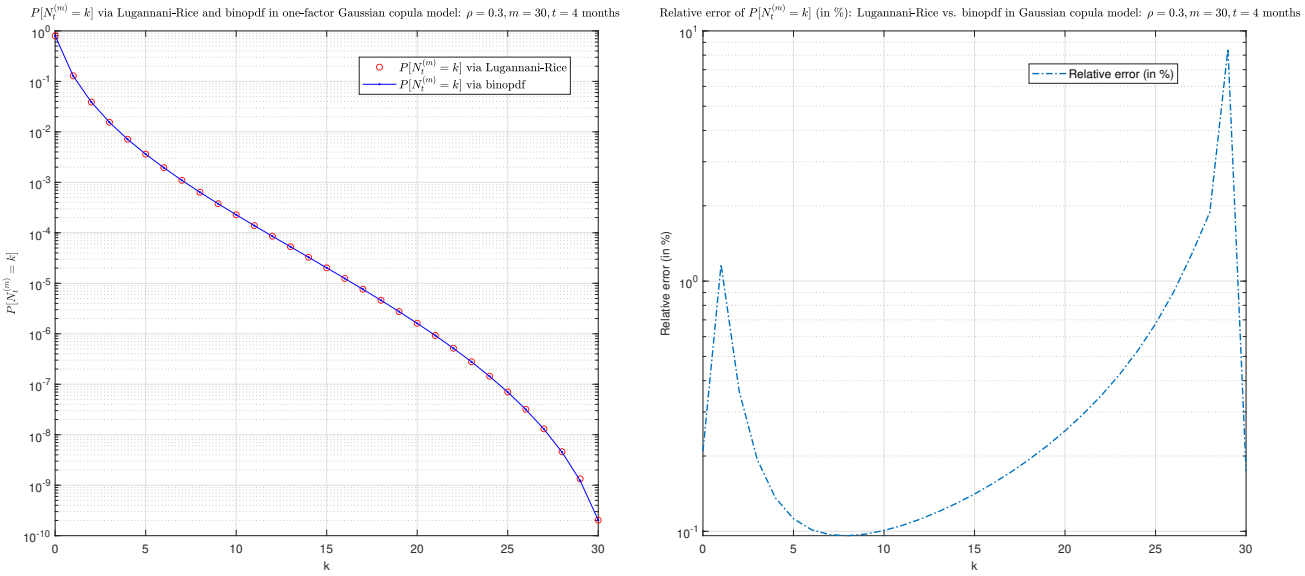


Figure 20. Left panel: Computation of $\mathbb{P}\left[N_t^{(m)} = k\right]$ both via the saddlepoint approach and Matlabs built in method binopdf where $k = 0, ..., m$ with $m = 30$ and $t = 4$ months, i.e. $t = \frac{4}{12}$, in a one-factor Gaussian copula model where $\rho = 0.3$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.0329$ (same parameters as in Table 2). **Right panel:** Relative error (in percent) between the two methods in the left panel where the relative difference is measured with respect to the binopdf-method.

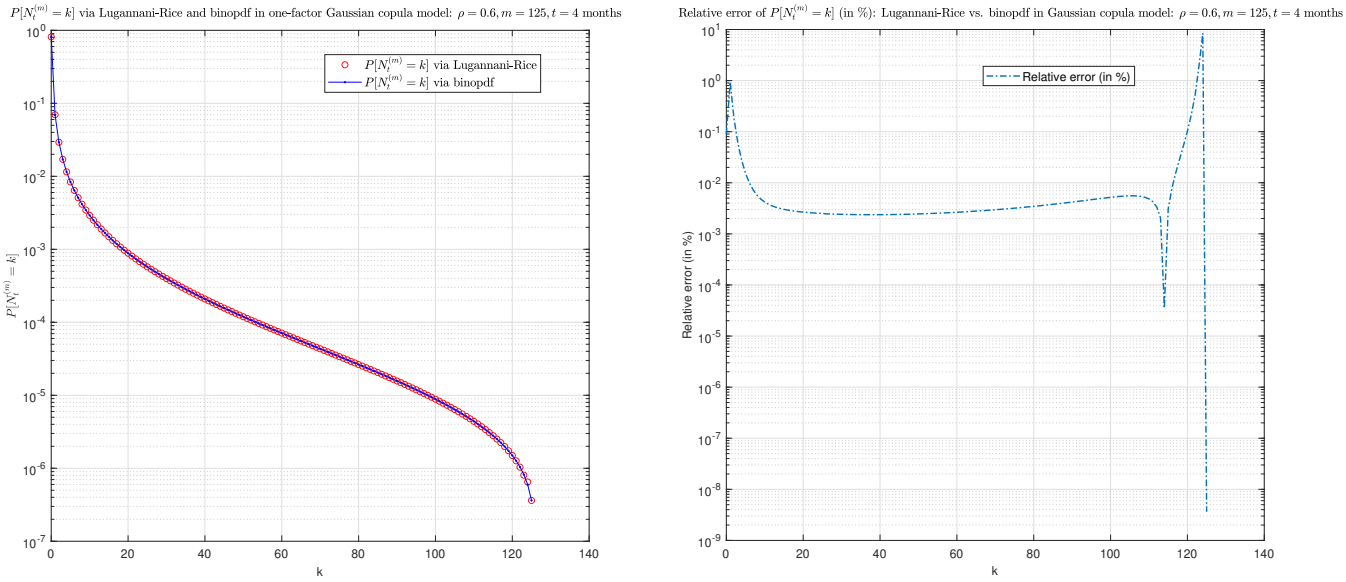


Figure 21. Left panel: Computation of $\mathbb{P}\left[N_t^{(m)} = k\right]$ both via the saddlepoint approach and Matlabs built in method binopdf where $k = 0, ..., m$ with $m = 125$ and $t = 4$ months, i.e. $t = \frac{4}{12}$, in a one-factor Gaussian copula model where $\rho = 0.6$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.0265$. **Right panel:** Relative error (in percent) between the two methods in the left panel where the relative difference is measured with respect to the binopdf-method.

In Figure 21 we repeat the same procedure as in Figure 20, but now for $t = 4$ months, i.e. $t = \frac{4}{12}$, in a one-factor Gaussian copula model where $\rho = 0.6$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.0265$. In the right panel of Figure 21 we see that the errors for $k = 0, ..., 122$ and $k = 125$ are all under 0.9454% while for $k = 123$ the error is 8.425% where $\mathbb{P}\left[N_t^{(125)} = 125\right] = 5.9830 \times 10^{-7}$ in the binopdf-case.

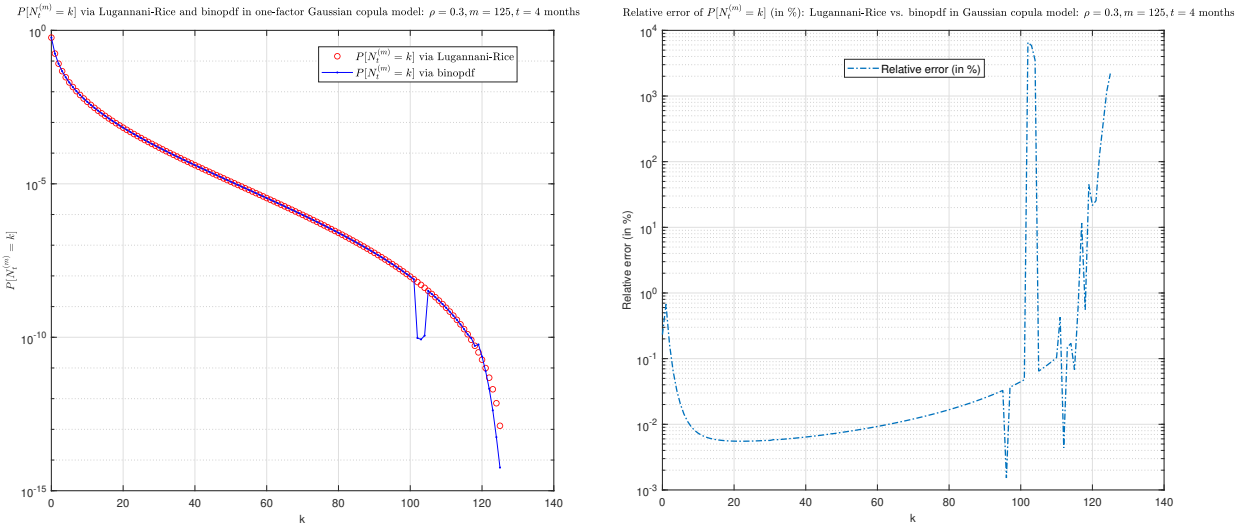


Figure 22. Left panel: Computation of $\mathbb{P}\left[N_t^{(m)} = k\right]$ both via the saddlepoint approach and Matlabs built in method binopdf where $k = 0, ..., m$ with $m = 125$ and $t = 4$ months, i.e. $t = \frac{4}{12}$, in a one-factor Gaussian copula model where $\rho = 0.3$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.0329$ (same parameters as in Table 2). **Right panel:** Relative error (in percent) between the two methods in the left panel where the relative difference is measured with respect to the binopdf-method.

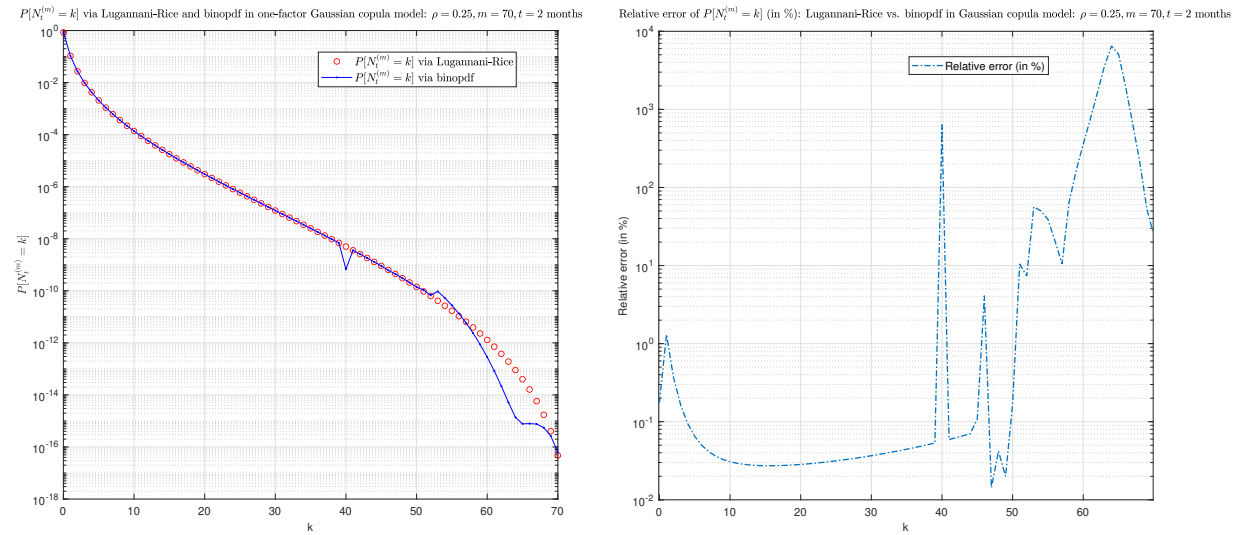


Figure 23. Left panel: Computation of $\mathbb{P}\left[N_t^{(m)} = k\right]$ both via the saddlepoint approach and Matlabs built in method binopdf where $k = 0, ..., m$ with $m = 70$ and $t = 2$ months, i.e. $t = \frac{2}{12}$, in a one-factor Gaussian copula model where $\rho = 0.25$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.02$. **Right panel:** Relative difference (in percent) between the two methods in the left panel where the relative difference is measured with respect to the binopdf-method.

Hence, in the cases shown both in Figure 20 and Figure 21 the saddlepoint approach gives an extremely good fit for all k, except for the second last points $k = m - 1$ both for $m = 30$ and $m = 125$, where the errors still are acceptable size (around 8.4%) and the probabilities are quite small at the points $k = m - 1$. We also remind that the saddlepoint method gives an upper bound of the error which is of order $O(m^{-3/2})$, see e.g. in Corollary (5.1) and Remark 5.2, but as can be seen in the left panels in Figure 20-21, most probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ are much lower than the value $m^{-3/2}$, and when comparing with the `binopdf`-approach the accuracy for the saddlepoint method for these cases are therefore much sharper than $m^{-3/2}$.

The results in Figure 20 - 21 are two examples where the alternative method via the `binopdf`-approach (11.3) actually works. However, in most of our numerical studies, the `binopdf`-approach will fail for some or many k-values. More specfic, we have found that, at least in the one-factor Gaussian copula model, it is more common that `binopdf`-approach in (11.3) will fail for various parameter settings, while the saddle point method in (11.4) seems to be remarkable robust. To see this, we repeat similar studies as Figure 20-21, but for other parameters not deviating too much from these in Figure 20-21. First, Figure 22 displays the case for $t = 4$ months, i.e. $t = \frac{4}{12}$, in a one-factor Gaussian copula model where $\rho = 0.3$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.0329$ and $m = 125$, which are the same parameters as in the Figures **??-??**. From the left panel in Figure 22, we see that the saddlepoint method produces a smooth curve for the probabilities, very similar to those in in Figure 20 - 21. However, the binopdf-method in Figure 22 gives a very irregular, and strange behaviour, for some k-values. Furthermore, from the right panel in Figure 22 we see that the relative error is very small (less than 0.6%) for $k = 0, \ldots, 101$ and $k = 105, \ldots, 110$ but for $k = 102, \ldots, 104$ and $k > 116$ the error jump up to several thousand percent, for example at $k = 102$ the error is 6549% and at $m = 125$ the error is 2200%. To this end, we also remind that the saddlepoint method will produce an exact computation at $k = m$, with zero error compared with the analytical binomial formula, implying that it is evident that the `binopdf`-approach fails substantially, at least for $k = 125$ in this case. Furthermore, the very strange irregularities seen in the left panel of Figure 22 for $k = 102, \ldots, 104$ in the binopdf-method, strongly indicates that the binopdf-method is not a stable numerical method, at least not in the one-factor Gaussian copula case. We have not been able to find out what goes wrong in the matlab binopdf-method for these values (and this is obviously not the purpose of our study), but they are clearly wrong, as can be seen in Figure 22. We also remind that Matlab does not provide a documentation of what method it uses in the routine "binopdf" to compute $\mathbb{P}\left[N_t^{(m)} = k\right]$ for larger values of m and k.

To show that failure of binopdf-method in Figure 22 is not isolated to one particular case, we redo the computations also for another case, where $m = 70$ and $t = 2$ months, i.e. $t = \frac{2}{12}$, in a one-factor Gaussian copula model where $\rho = 0.25$ and $F(t) = 1 - e^{-\lambda t}$ with $F(1) = 0.02$, and display the results in Figure 23. Note that the irregularities/failures by the matlab binopdf-method are in these case even more pronounced. For example, as seen in the right panel of Figure 23, the relative error is very small (less than 1.31%) for $k = 0, \ldots, 39$ and also small errors for $k = 41, \ldots, 50$ (less than 4.11%)), but for $k = 40$ we suddenly have a dramatic jump of the error to more than 637% which clearly is due to some flaw of the binopdf-approach, as can be seen in the left panel of Figure 23, since it is rather smooth for all k-values in a neighborhood of $k = 40$. Also note the very strange deviation of the binopdf-method for $k \geq 53$ with an error of over 6485% at $k = 64$. Finally, for $k = 70$ the error is 24.75% while it should be zero if binopdf works properly, since at $k = m$ the saddlepoint gives an exact expression for $\mathbb{P}\left[N_t^{(m)} = m\right]$, as mentioned above, showing once again that binopdf fails to work for mixed binomial models, as in (11.2)-(11.3).

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