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RISK MANAGEMENT OF STOCK PORTFOLIOS WITH JUMPS AT EXOGENOUS DEFAULT EVENTS

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Abstract. In this paper, we study equity risk management of stock portfolios where the individual stock prices have downward jumps at the defaults of an exogenous group of defaultable entities. The default times can come from any type of credit portfolio model. In this setting, we derive computational tractable formulas for several stock-related quantizes, such as loss distributions of equity portfolios, and apply it to Value-at-Risk computations. In the portfolio case, our study considers both small-time expansions of the loss-distribution for a heterogeneous portfolio via a linearization of the loss, but also for general time points when the stock portfolio is large and homogeneous and where we use a conditional version of the law of large numbers. Most of the derived formulas will heavily rely on the ability to efficiently compute the number of defaults distribution of the entities in the exogenous group of corporates negatively affecting the stock prices in our equity portfolio. We give several numerical applications. For example, in a setting where the jumps in the stock prices are at default times which are generated by a one-factor Gaussian copula model, we study the time evolution of Value-at-Risk (i.e. VaR as function of time) for stock portfolios, both for a 20-day period and for a two-year period. We also perform similar numerical VaR-studies in a setting where the individual default intensities follow a CIR process. Our results are compared with the corresponding VaR-values in both the Black-Scholes case and Kou model with the same drift and volatility as in the jump-at-default models. Unsurprisingly, we show that the VaR-values in stock portfolios with downward jumps at defaults of external entities will have substantially higher VaR-values compared to the corresponding Black-Scholes cases, but also compared with VaR-numbers from the Kou model restricted to having only negative jumps.

1. Introduction.

Simultaneous downward jumps in multiple stock prices at defaults of large companies is a very realistic feature. For example, at the default of Lehmann Brothers on September 15th, 2008, there was a 4.5% drop in the Dow-Jones Industrial Average index during the trading day, while the S&P 500 jumped down nearly 5% the same day, see e.g. in [31]. Other more recent examples are the

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collapse of Silicon Valley Bank in March 2023 which caused a 6.6% and a 3.8% one-day declines in the S&P 500 banking index and Europe's STOXX banking index respectively, see [46].

In this paper, we study equity risk management of stock portfolios where the individual stock prices have simultaneous downward jumps at the defaults of an exogenous group of defaultable entities, for example corporate or sovereign states. By "exogenous", here we mean that the entities, for example companies, will not be represented in the stock portfolio, that is stocks issued by the defaultable corporates are not present in the stock portfolio in our studies. The default times can come from any type of credit portfolio model. In this setting, we derive computational tractable formulas to several stock-related quantizes, for example the loss distributions of equity portfolios, and apply it to risk management computations, such as Value-at-Risk of portfolios. We start with modeling an individual stock price and derive expressions for the expected value, conditional expected value, density, and distribution for the stock. In the stock portfolio case, our study considers both small-time expansions of the loss-distribution to a heterogeneous portfolio via a linearization of the loss, but also for general time points when the stock portfolio is large and homogeneous, where we utilize a conditional version of the law of large numbers for a homogeneous stock portfolio. Most of the formulas in this paper will heavily rely on the ability to efficiently compute the number of defaults distribution of the entities in the exogenous group which are negatively affecting the stock prices in our equity portfolio. In the case when the stock prices are unaffected by the exogenous defaults, our stock price model collapses into the traditional Black-Scholes model under the real probability measure. Finally, we give several numerical applications. For example, in a setting where the jumps in the stock prices are at default times which are generated by a one-factor Gaussian copula model, we study the time evolution of Value-at-Risk (i.e. VaR as function of time) for stock portfolios, both for a 20-day period with one-day steps and for a two-year period with one-month steps. In the 20-day period we use the linear approximation for the loss-process, while for the two-year period we utilize a large portfolio approximation formula for the loss-process to a large homogeneous stock portfolio. We also perform similar studies when the default times are generated by a Clayton copula model. Furthermore, in a setting where the jumps in the stock price are at default times which have CIR-intensities, we also study the time evolution of Value-at-Risk for one stock over a two-year period. In all our numerical computations, we compare our results with the corresponding VaR-values in the Black-Scholes case with the same drift and volatility as in the jump models and in some cases also with the [32] model restricted to only having negative jumps with suitable chosen jump parameters to make the comparison fair. Unsurprisingly, we show that the VaR-values in stock portfolios with stock that jump downward at defaults of external companies will have substantially higher VaR-values compared to the corresponding Black-Scholes cases, and also much higher than those VaR-metricis coming from the [32] model with only negative jumps. The numerical computations of the number of default distributions will in all our VaR-studies use fast and efficient saddlepoint methods developed in [25].

There exists a huge amount of academic papers that model stock prices with jumps, and a vast majority of these articles which contains numerical/practical examples consider the case where jump times are driven by some sort of Poisson process. Furthermore, most of the jump-related equity papers model the stock price directly under the risk neutral probability measure and then apply the model for option pricing, such as, e.g., the original paper by [41]. An example of an article that actually models the stock price under the real (physical) probability measure is the seminal paper [32], where the stock price jump either up or down at random times driven by a Poisson process with constant intensity. [32] mainly studied option pricing directly under the real probability measure by using asset pricing theory, consumption utilization, and the Euler equation, where both the endowment process and the stock price follows the type of jump diffusion as defined in [32]. More about option pricing models for stocks with jumps driven by Poisson processes (such as Levy processes) can be found in, e.g., the books [48] or [12].

In this paper, all jumps in the stock prices are downward jumps occurring at the defaults in an exogenous group of defaultable entities. Hence, in this paper we have explicitly inserted "external" credit risk (from the external group of defaultable entities) into the equity dynamics or our stock price, effectively creating a type of *hybrid risk model*. Thus, the stock price model in this paper involves *both equity and credit risk*, although the credit risk comes from an external group of defaultable entities which can be corporate or sovereign states. Furthermore, we work under the real (physical) probability measure and focus on risk management, such as VaR computations of stock portfolios. To the best of our knowledge, this is the first paper that numerically computes VaR and related risk management quantities for stock portfolios where all the stock prices have simultaneous jumps at defaults in an external group of arbitrarily many defaultable entities.

Assuming only negative jumps in the stock prices will lead to a more conservative or prudent equity portfolio model which implies larger Value-at-Risk losses compared to a model which also includes positive jumps. Including only negative jumps in stock prices for, e.g., VaR-models should therefore be more favorable among financial regulators (such as e.g. SEC, FCA, BaFin etc.) compared with frameworks that also contain positive jumps in equity prices. In our model, it is possible to add another jump process in the dynamics of the stock price, for example a Poisson process with constant intensity and with positive jumps, e.g., as in [32]. However, in this paper we are *only* interested in studying the effect of *external credit risk* on stock prices coming from the external group of defaultable entities, and therefore our jump-part in the dynamics of the stock will only include negative jumps occurring at the external default times. Furthermore, if the defaultable entities used in our stock price model have issued bonds (or stocks) which are publicly traded on major financial markets, then typically their default times are exogenously observed, as for example the default of Lehmann Brothers in 2008. On the other hand, if a Poisson process drives the times when the stock price jumps, then these jump times can be difficult to observe exogenously and also difficult to assign to a specific financial event.

We want to emphasize that we in this paper do not focus on how to estimate the involved parameters describing our stock model, including the parameters for the defaultable entities affecting the equity prices. Instead, the main goal of this paper is to derive analytical stock portfolio quantities in our equity-credit hybrid model and then use these to numerically study the time evolution of VaR for equity portfolios and compare the VaR numbers with corresponding values coming from alternative models, such as the Kou model and the Black-Scholes model. The topic of estimating parameters in stock price models with jumps under the real probability measure is a complex problem, see for example [16,33] and [37], which all focus on jumps coming from Poisson processes and not from exogenous defaults as in this paper. Furthermore, since the number of defaults of large companies are scarce, estimation of the parameters for the defaultable entities under the physical probability measure is also challenging and in fact a completely separate credit risk problem not connected to our stock price model. However, to get a better understanding of how VaR-values in our stock price model are affected by the credit parameters, [25] investigated stock portfolio VaR as function of some of the parameters describing the defaultable entities, such as the one-factor Gaussian copula correlation parameter when it runs through an interval on the positive real line bounded by one.

The rest of this paper is organized as follows. First, in Section 2 we consider one stock where the stock price can jump at default times belonging to an exogenous group of defaultable entities, and then derive all relevant quantities, such as the expected value, conditional expected value, density, and distribution both for the stock and its loss process. Next, in Section 3 we prove that the stock price model in this paper can never be equal in distribution with a model where the m default times are replaced with the m first jumps of some Cox process, in particular not a Markov-modulated Poisson process, where m is the number of defaultable entities affecting the stock price. Hence, the model presented in this paper is unique in the sense that it can not be seen as a special case of, e.g., the papers [9] or [32], or any other model based on [32] where the Poisson process is replaced with a Cox process and where all jumps in the stock price are negative and have the same distribution. In Section 4, we generalize the single-stock dynamics in Section 2 to a heterogeneous portfolio of stocks and define the loss process for the stock portfolio. Furthermore, for small time points we make a linearization of the portfolio loss process and then derive a computationally tractable expression for the distribution of the linearized loss. For larger time points t, the linear approximations to the stock portfolio in Section 2 will fail, but in Section 5 we outline a method that will work for arbitrary time points for large homogeneous stock portfolios and derive a convenient expression for the distribution of the portfolio loss in such settings by using large portfolio approximations. In Section 6, we present a version of the [32] model restricted to only having negative jumps and also derive some important quantities which will be used in the numerical section were we compare stock portfolio VaR-values in this Kou model with corresponding VaR-numbers coming from an equity model with jumps at defaults outlined in Section 5. In the numerical part of the paper covered in Section 7-10, we give several practical applications of our developed stock price model. First, in Section 7 we study Value-at-Risk over a two-year period for the loss of one single stock when the stock price is defined as in Section 2 in a model where the default times are exchangeable, conditionally independent, and have CIR-intensities. In Section 8, we repeat similar VaR-studies as in Section 7, but now for a portfolio of stocks in a setting with jumps in all stock prices occurring at default times driven by a one-factor copula model, and by using the small-time expansion formulas for the loss process derived in Section 4. In the Gaussian copula model, in Section 8 we also do VaR computations for large stock portfolios by using the large portfolio approximation formulas derived in Section 5, both for a 20-day period in time steps of one trading day, but also over a two-year period in time steps of one month. In Section 9, we repeat the same type of studies as in Section 8, but now for a Clayton copula in the large homogeneous stock portfolio case. All computations done in Sections 7-9 heavily rely on efficient numerical methods developed in [25] for computing the distribution of the number of defaults among the defaultable entities creating the jumps in the stock prices. Finally, in Section 10 we compute VaR for a stock portfolio model derived from the [32] model, restricted to only having negative jumps, as outlined in Section 6, and then compare these VaR-values with the corresponding VaR-metrics coming from our jump-at-defaults model for a one-factor Gaussian copula model as outlined in Section 5.

2. **The one-dimensional case.** In this subsection, we consider one stock where the stock price can jump at default times belonging to an exogenous group of defaultable entities. We first define the dynamics of the stock price under the real (physical) probability measure $\mathbb{P}$ that will be used throughout the first sections of the paper. Furthermore, we also derive all relevant quantities for the single stock, such as the expected value, conditional expected value, density, and distribution both for the stock and its loss process. We start with the following definition of the stock price.

Definition 2.1. Consider a group of m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ with individual default times $\tau_1, \tau_2 \dots, \tau_m$, and let $\tilde{V}_1, \dots, \tilde{V}_m$ be random variables which have bounded expected values, satisfy $\tilde{V}_i \geq -1$, and are independent of $\tau_1, \tau_2 \dots, \tau_m$. Let company $\mathbf{A}$ be an entity which does **not belong** to the group $\mathbf{C}_1, \dots, \mathbf{C}_m$, and let S_t denote the price of the stock to company $\mathbf{A}$ at time t. The dynamics of S_t under the real probability measure $\mathbb{P}$ is defined as

$$dS_t = S_{t-} dY_t \tag{2.1}$$

where Y_t is given by

$$Y_t = \mu t + \sigma W_t + \sum_{i=1}^{m} \tilde{V}_i 1_{\{\tau_i \leq t\}} \tag{2.2}$$

and W_t is Brownian motion independent of the default times $\tau_1, \tau_2 \dots, \tau_m$ and $\tilde{V}_1, \dots, \tilde{V}_m$. Finally, $\sigma \geq 0$ is the so called volatility and μ is denoted as the drift of the stock price S_t.

Remark 2.2. We remark that the default times $\tau_1, \tau_2 \dots, \tau_m$ in Definition 2.1 can come from any credit portfolio model as long as the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ in the stock prices at the default times $\tau_1, \tau_2 \dots, \tau_m$ are independent of these defaults and also independent of the Brownian motion. We can, for example work, with heterogeneous or homogeneous copula based models studied in, e.g., [1, 8, 14, 19, 29, 35, 36], or heterogeneous or homogeneous conditional independent intensity based models, such as in [3–5], and [2], as well as heterogeneous or homogeneous contagion models studied in, e.g., [10, 11, 17, 20–24, 27, 34], and [18].

Remark 2.3. Relation to the model [32]. Note that the stock price S_t in Definition 2.1 is related to the seminal paper [32]. The main difference between [32] and Definition 2.1 is that [32] considers jumps coming from a Poisson process with constant intensity, implying possibly infinity many jumps, while the jumps in Definition 2.1 are due to the default times $\tau_1, \tau_2 \dots, \tau_m$, which comes from a finite group of m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. Hence, Definition 2.1 implies that at each default time τ_i among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$, the stock price S_t will jump so that $\Delta S_{\tau_i} \neq 0$ and the jump-times of S_t therefore have a direct financial interpretation, namely the default times τ_i among the firms $\mathbf{C}_1, \dots, \mathbf{C}_m$. Hence, the major difference between S_t in Definition 2.1 in this paper and the model by [32] is that in Definition 2.1 we have explicitly inserted "external" credit risk (from the external group $\mathbf{C}_1, \dots, \mathbf{C}_m$) into the equity dynamics for S_t, effectively creating a type of *hybrid risk model*, that is, the stock price model S_t involves *both equity and credit risk*, although the credit risk comes from an external group of m entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$. In Definition 2.1, it is possible to add another jump process in the dynamics of S_t, for example a Poisson process with constant intensity which jumps just as in [32]. However, in this paper we are *only* interested in studying the effect of *external credit risk* on S_t coming from the external group of defaultable entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$, and therefore our jump-part in the dynamics of S_t will only include the jumps coming from the default times $\tau_1, \tau_2 \ldots, \tau_m$ of $\mathbf{C}_1, \ldots, \mathbf{C}_m$. Another remark is that [32] mainly studies option pricing directly under the real measure $\mathbb{P}$ by using the Euler equation where both the endowment process and the stock price follows the type of jump diffusion as given in Section 2 of [32], and where the utility function has the special form $U(c,t) = e^{-\theta t}\frac{c^\alpha}{\alpha}$ for $0 < \alpha < 1$ or $U(c,t) = e^{-\theta t}\ln c$ for $\alpha = 0$. In this paper, we will focus on equity risk management of stock portfolios (such as Value-at-Risk) where the individual stock prices have downward jumps down at the defaults of an exogenous group of defaultable entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$, as given in Definition 2.1, and we will consider both univariate and multivariate stock portfolios, as well as the case where the number of stocks in the portfolio is large. In our Value-at-Risk studies of the stock portfolios, we are in particularly interested in studying the effect of *external credit risk* coming from the external defaultable group of entities.

Finally, we remark that if the defaultable entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ have issued bonds and/or stocks which are publicly traded on major financial markets, then typically the default times $\tau_1, \tau_2 \ldots, \tau_m$ are *directly observable* on the market, and the observations are exogenously observed regardless of if the stock price model for S_t includes these defaults or not. This has to be compared with if a Poisson process drives the jumps which can be difficult to observe exogenously and also difficult to assign to specific financial events.

Remark 2.4. On the possiblity to include company $\mathbf{A}$ in the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$. We remark that, in Definition 2.1, it is possible to let company $\mathbf{A}$ be one of the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$, for example $\mathbf{A} = \mathbf{C}_m$ where we then set $\tilde{V}_m = -1$ so that $S_t = 0$ for $\tau_m \leq t$ where τ_m is the default time of $\mathbf{A}$. Including $\mathbf{A}$ in the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$ where, e.g., $\mathbf{A} = \mathbf{C}_m$ will create an extra complexity in the stock-related formulas, in particular if the default time of $\mathbf{A}$ will be correlated with the default times of $\mathbf{C}_1, \ldots, \mathbf{C}_{m-1}$. However, in this paper we are *only* interested in studying the effect of *external credit risk* coming from the external defaultable group of entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ (for example when studying how the external credit risk affect Value-at-Risk for S_t), and we will therefore in this paper always assume that company $\mathbf{A}$ will *not belong* to the defaultable group $\mathbf{C}_1, \ldots, \mathbf{C}_m$.

Remark 2.5. Stochastic volatility. In the dynamics of Definition 2.1, it is possible to replace the constant volatility with, e.g., a stochastic volatility, as in the Heston model presented in [28]. However, allowing for a stochastic volatility such as in [28] will no longer lead to closed-formulas for the stock price dynamics or semi-closed formulas for the portfolio loss distribution. Instead, we have to rely on Fourier inversion techniques (e.g. FFT-methods) to find the distribution of the stock portfolio losses, which will make the Value-at-Risk computations much more time-consuming. Therefore, will in this paper not consider stochastic volatilities in the stock price model given by Definition 2.1.

We now state the following useful proposition which is proved in Subsection A.1 of Appendix A.

Proposition 2.6. *Let S_t be a stock price given by Definition 2.1 under the real probability measure $\mathbb{P}$. Then, with notation as above, we have*

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) \prod_{i=1}^{m} \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}}\right) . \tag{2.3}$$

Let $\tilde{V}_i$ be the random variable in Definition 2.1 connected to default of company $\mathbf{C}_i$ at the random default time τ_i in Definition 2.1. Then, Proposition 2.6 implies that for any default time τ_i among the m entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$, we have that

$$S_{\tau_i} = S_{\tau_i -}\left(1 + \tilde{V}_i\right) \quad \text{or equivalently} \quad \frac{S_{\tau_i} - S_{\tau_i -}}{S_{\tau_i -}} = \tilde{V}_i$$

i.e., there is a relative jump of random size $\tilde{V}_i$ of the stock price S_t to company $\mathbf{A}$ at the default time τ_i of entity $\mathbf{C}_i$, where we remind that $\tilde{V}_i \geq -1$.

Note that if there are no jumps at the defaults of $\mathbf{C}_1, \ldots, \mathbf{C}_m$, that is, if $\tilde{V}_n = 0$ for all k in Definition 2.1, then (2.3) in Proposition 2.6 implies that we are back in the classical Black-Scholes model under the real (physical) probability measure $\mathbb{P}$, with drift μ and volatility σ, that is

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) . \tag{2.4}$$

In the paper [32], the jumps $\tilde{V}_i$ can be both positive or negative, where the jumps occur at the arrivals of a Poission process, implying that the stock price can jump both up and down. In this paper, we will consider all defaults among the m entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ in Definition 2.1 as negative news for company $\mathbf{A}$, implying that the relative jumps $\tilde{V}_i$ of the stock price S_t to company $\mathbf{A}$ at each default of $\mathbf{C}_1, \ldots, \mathbf{C}_m$ will be negative. Hence, in this paper the stock price S_t will jump downwards at the default times $\tau_1, \tau_2 \ldots, \tau_m$. Furthermore, we define $\tilde{V}_i$ as follows.

Definition 2.7. Let $\tilde{U}_1, \ldots, \tilde{U}_m$ be arbitrary non-negative random variables which have bounded expected values, are independent of the default times $\tau_1, \tau_2 \ldots, \tau_m$ and are also independent of W_t in Definition 2.7. Then, we define the negative random variables $\tilde{V}_1, \ldots, \tilde{V}_m$ as

$$\tilde{V}_i = e^{-\tilde{U}_i} - 1 \tag{2.5}$$

for each defaultable entity $\mathbf{C}_1, \ldots, \mathbf{C}_m$.

From (2.5), it is easy to see that

$$\tilde{V}_i 1_{\{\tau_i \leq t\}} = \exp\left(-\tilde{U}_i 1_{\{\tau_i \leq t\}}\right) - 1 \quad \text{for all } t \geq 0$$

so that

$$\prod_{i=1}^{m} \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}}\right) = \exp\left(-\sum_{i=1}^{m} \tilde{U}_i 1_{\{\tau_i \leq t\}}\right) . \tag{2.6}$$

Hence, in view of Definition 2.10 and Equation (2.6), we state the following corollary to Proposition 2.6.

Corollary 2.8. *Let S_t be a stock price given by Definition 2.1 under the real probability measure $\mathbb{P}$ and where the jumps $\tilde{V}_1, \dots, \tilde{V}_m$ are distributed as in Definition 2.7 via the arbitrary non-negative random variables $\tilde{U}_1, \dots, \tilde{U}_m \in L_1$. Then, with notation as above, we have*

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{i=1}^{m} \tilde{U}_i 1_{\{\tau_i \le t\}}\right). \tag{2.7}$$

In this paper, we are primary interested in finding computationally tractable expressions for the distribution of the stock price S_t, and use this distribution in various risk management applications under the real probability measure $\mathbb{P}$. Under Definition 2.1 and Definition 2.7 with heterogeneous distributions for $\tilde{U}_1, \dots, \tilde{U}_m$, it is then clear from Corollary 2.8 that the distribution of the stock price S_t will be a sum containing up to 2^m different terms. Furthermore, to find the $\mathbb{P}[S_t \le x]$ we need, for each set of defaultable entitles $\mathbf{i}_k = (i_1, \dots, i_k)$, $\mathbf{i}_k \subseteq \{1, \dots, m\}$ among the group $\mathbf{C}_1, \dots, \mathbf{C}_m$, to be able to find expressions for the distribution of $\sum_{n=1}^{k} \tilde{U}_{i_n}$. Note that there are $\binom{m}{k}$ different ways to pick out a subset $\mathbf{i}_k \subseteq \{1, \dots, m\}$ such that $\mathbf{i}_k = (i_1, \dots, i_k)$, which represents the defaults of the k entities $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_k}$ among the m entities $\mathbf{C}_1, \dots, \mathbf{C}_m$, and where the ordering of $i_1, \dots, i_k$ is ignored. The ordering of how the group $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_k}$ defaults is not important, explaining the term $\binom{m}{k}$ compared to the case where ordering matters, which leads to $k!\binom{m}{k}$ different ways to pick out $\mathbf{i}_k$. The reason why we can ignore the ordering of the defaults follows from the structure of the jumps in (2.7) in Corollary 2.8 where we only need to keep track of *if* an entity $\mathbf{C}_i$ has defaulted or not. Thus, the total number of possible distinct terms in the expression for $\mathbb{P}[S_t \le x]$ will be

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m .$$

For example, if $m = 15$ with m different distributions of $\tilde{U}_1, \dots, \tilde{U}_m$, this will then lead to up to possibly $2^{15} = 32768$ different terms in the distribution $\mathbb{P}[S_t \le x]$. These observations makes the definition of the stock price S_t in Definition 2.1 and Definition 2.7 with heterogeneous distributions for $\tilde{U}_1, \dots, \tilde{U}_m$ unusable from a practical point of view, even for moderate sizes m of the group of entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ that affect the stock price.

However, if $\tilde{U}_1, \dots, \tilde{U}_m$ are exchangeable, for example if $\tilde{U}_1, \dots, \tilde{U}_m$ is an i.i.d sequence and thus are homogeneous in their distributions, then the number of terms in the sums for $\mathbb{P}[S_t \le x]$ will reduce from 2^m to just m terms, which will be practical to handle for very large m-values, such as $m > 100$ entities in the group $\mathbf{C}_1, \dots, \mathbf{C}_m$. To see why the terms reduce from 2^m to m, let $N_t^{(m)}$ be a point process that counts the number of defaults among the m defaultable entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ with default times $\tau_1, \tau_2 \dots, \tau_m$, that is

$$N_t^{(m)} = \sum_{i=1}^{m} 1_{\{\tau_i \le t\}} . \tag{2.8}$$

Furthermore, if $\tilde{U}_1, \dots, \tilde{U}_m$ is an i.i.d sequence and if $U_1, \dots, U_m$ is another i.i.d sequence with the same distribution as $\tilde{U}_1, \dots, \tilde{U}_m$, then we have that

$$\sum_{i=1}^{m} \tilde{U}_i 1_{\{\tau_i \le t\}} \stackrel{d}{=} \sum_{n=1}^{N_t^m} U_n \tag{2.9}$$

where $N_t^{(m)}$ is defined as in (2.8), so Corollary 2.8 and (2.9) therefore imply that

$$S_t \overset{d}{=} S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n\right) \tag{2.10}$$

where we remind that for two random variables X and Y, the notation $X \overset{d}{=} Y$ means that X and Y have the same distribution. In view of Equation (2.9)-(2.10), we will sometimes use the notation $U_1, \ldots, U_m$ and $V_1, \ldots, V_m$ instead of $\tilde{U}_1, \ldots, \tilde{U}_m$ and $\tilde{V}_1, \ldots, \tilde{V}_m$, and sometimes write $S_t = ...$ instead of $S_t \overset{d}{=} ...$ in Equation (2.10).

Remark 2.9. The reason why the exchangeability of the jumps $\tilde{U}_1, \ldots, \tilde{U}_m$ are important is that, if this is not true, we have to keep track of *which* of the companies $\mathbf{C}_1, \ldots, \mathbf{C}_m$ have defaulted up to time t, while in the exchangeability case for $\tilde{U}_1, \ldots, \tilde{U}_m$ we only need to keep track of *how many* of $\mathbf{C}_1, \ldots, \mathbf{C}_m$ that have defaulted up to time t, i.e. we only need to model $N_t^{(m)}$ defined as in (2.8).

Next, we make the following assumption on $U_1, \ldots, U_m$ and $V_1, \ldots, V_m$.

Definition 2.10. Let $U_1, \ldots, U_m$ be an i.i.d sequence of exponentially distributed random variables which are independent of W_t and also independent of the default times $\tau_1, \tau_2 \ldots, \tau_m$. Then, we define the i.i.d sequence $V_1, \ldots, V_m$ as

$$V_n = e^{-U_n} - 1 \quad \text{where} \quad U_n \overset{d}{=} \text{Exp}(\eta) \quad \text{with} \quad \mathbb{E}\left[U_n\right] = \frac{1}{\eta}. \tag{2.11}$$

From (2.11) in Definition 2.1, we see that U_n is exponentially distributed with density $\eta e^{-\eta u}$ for $u \geq 0$, and that $V_n \geq -1$ for each n.

Remark 2.11. Note that if $\eta \to \infty$, then $U_n \to 0$ almost surely under $\mathbb{P}$, so with a slight abuse of notation, we can identify $U_n = 0$ with "$\eta = \infty$".

The definition in (2.11) is similar to the one on p.1087 in [32], but where we here restrict ourselves to only negative jumps in stock price, while [32] allows for both positive and negative stock price jumps. Assuming only negative jumps as in our model will lead to a more conservative or prudent stock price model which in particular will lead to larger Value-at-Risk losses, and should therefore be more favourable among financial regulators (such as e.g. SEC, FCA, BaFin etc.) compared with models that also includes positive jumps in stock prices.

Remark 2.12. Note that our choice of U_i in Definition 2.1 as an exponentially distributed variable will together with the exchangeability condition lead to analytical expressions for the distribution of S_t, but also for the stock portfolio losses studied in Section 4 and Section 5. Another choice of U_i leading to analytical expressions for the portfolio loss distributions is a normal distribution, see e.g in [41]. However, sums of i.i.d normal distributions will have a non-zero probability of attaining positive values, even though the individual means are negative. Note that having positive jumps in stocks at defaults of large corporates is very unusual. For example, at the default of Lehmann Brothers on September 15th, 2008, there was a 4.5% drop in the Dow-Jones Industrial Average index during the trading day, while the S&P 500 jumped down nearly 5% the same day, see in [31].

In view of Definition 2.10 and Equation (2.10), we state the following corollary to Proposition 2.6.

Corollary 2.13. *Let S_t be a stock price given by Definition 2.1 under the real probability measure $\mathbb{P}$ where the jumps $\tilde{V}_1, \ldots, \tilde{V}_m$ are distributed as $V_1, \ldots, V_m$ in Definition 2.10 with $\eta > 0$. Then, with notation as above,*

$$S_t \stackrel{d}{=} S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n\right). \tag{2.12}$$

Next, define the loss process $L_t^{(S)}$ for the stock S_t at time t with reference to the starting time 0, as

$$L_t^{(S)} = -(S_t - S_0) \tag{2.13}$$

where we note that a gain implies that the loss $L_t^{(S)}$ is negative. We are interested in computing Value-at-Risk for $L_t^{(S)}$ in our model for a stock price with jumps at defaults, that is, we want to compute

$$\mathrm{VaR}_\alpha\left(L_t^{(S)}\right) = \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[L_t^{(S)} > y\right] \leq 1-\alpha\right\} = \inf\left\{y \in \mathbb{R} : F_{L_t^{(S)}}(y) \geq \alpha\right\} \tag{2.14}$$

where $F_{L_t^{(S)}}(x)$ is the distribution of $L_t^{(S)}$ and α is the confidence level, typically given by $95\%, 99\%$, or 99.9%, that is, $\alpha = 0.95, \alpha = 0.99$, or $\alpha = 0.999$. So, if S_t is given as in Definition 2.1 with jumps as in Definition 2.10, then in view of Corollary 2.13, the loss $L_t^{(S)}$ in (4.9) can be reformulated as

$$L_t^{(S)} \stackrel{d}{=} S_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n\right)\right) \tag{2.15}$$

where for any $t > 0$ we have $\sup L_t^{(S)} = S_0$, since $S_t \geq 0$ almost surely. Next, we state the following useful theorem which is proved in Subsection A.2 of Appendix A.

Theorem 2.14. *Let S_t be a stock price under the real probability measure $\mathbb{P}$ defined as in Corollary 2.13. Then, with notation as above, we have that*

$$\mathbb{E}\left[S_t \,|\, N_t^{(m)}\right] = S_0 e^{\mu t}\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \quad \text{where} \quad \mathbb{E}\left[S_t \,|\, N_t^{(m)} = k\right] = S_0 e^{\mu t}\left(\frac{\eta}{\eta+1}\right)^k \tag{2.16}$$

for $k = 0, 1, 2, \ldots, m$ and

$$\mathbb{E}[S_t] = S_0 e^{\mu t}\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right] = S_0 e^{\mu t}\sum_{k=0}^{m}\left(\frac{\eta}{\eta+1}\right)^k \mathbb{P}\left[N_t^{(m)} = k\right]. \tag{2.17}$$

Furthermore,

$$\mathbb{P}[S_t \leq x] = \sum_{k=0}^{m} \Psi_k(x, t, \mu, \sigma, S_0, \eta)\,\mathbb{P}\left[N_t^{(m)} = k\right] \tag{2.18}$$

where the mappings $\Psi_k(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ are defined as

$$\begin{aligned}&\Psi_k(x, t, \mu, \sigma, u, \eta)\\ &= \int_0^\infty \Phi\left(\frac{\ln\frac{x}{u} - \left(\mu - \frac{1}{2}\sigma^2\right)t + y}{\sigma\sqrt{t}}\right)\frac{\eta e^{-\eta y}(\eta y)^{k-1}}{(k-1)!}\,dy \quad \text{for} \quad 0 < k \leq m\end{aligned} \tag{2.19}$$

and $\Psi_0(x,t,\mu,\sigma,u,\eta)$ *for* $u>0$ *is given by*

$$\Psi_0(x,t,\mu,\sigma,u,\eta)=\Phi\left(\frac{\ln\frac{x}{u}-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \tag{2.20}$$

where $\Phi(x)$ *is the distribution function of a standard normal random variable. Furthermore,*

$$F_{L_t^{(S)}}(x)=\mathbb{P}\left[L_t^{(S)}\leq x\right]=1-\sum_{k=0}^{m}\Psi_k\left(1-\frac{x}{S_0},t,\mu,\sigma,1,\eta\right)\mathbb{P}\left[N_t^{(m)}=k\right] \tag{2.21}$$

where $x\leq S_0$*, and for any* $t>0$*, we have* $\sup L_t^{(S)}=S_0$*. The density* $f_{S_t}(x)$ *of* S_t *is given by*

$$f_{S_t}(x)=\sum_{k=0}^{m}\psi_k(x,t,\mu,\sigma,S_0,\eta)\,\mathbb{P}\left[N_t^{(m)}=k\right]\qquad \text{for } x>0,t>0 \tag{2.22}$$

where the mappings $\psi_k(x,t,\mu,\sigma,S_0,\eta)$ *for* $S_0>0,x>0,andt>0$ *are defined as*

$$\begin{aligned}&\psi_k(x,t,\mu,\sigma,S_0,\eta)\\&=\frac{1}{x\sigma\sqrt{t}}\int_0^\infty\varphi\left(\frac{\ln\frac{x}{S_0}-\left(\mu-\frac{1}{2}\sigma^2\right)t+y}{\sigma\sqrt{t}}\right)\frac{\eta e^{-\eta y}(\eta y)^{k-1}}{(k-1)!}\,dy\quad \text{for}\quad 0<k\leq m\end{aligned} \tag{2.23}$$

and $\psi_0(x,t,\mu,\sigma,S_0,\eta)$ *for* $S_0>0$ *is given by*

$$\psi_0(x,t,\mu,\sigma,S_0,\eta)=\frac{1}{x\sigma\sqrt{t}}\varphi\left(\frac{\ln\frac{x}{S_0}-\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \tag{2.24}$$

where $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ *is the density of a standard normal random variable.*

A full proof of Theorem 2.14 is given in Subsection A.2 of Appendix A.

We now make some remarks connected to Theorem 2.14.

Remark 2.15. As pointed out in Remark 2.3, if the defaultable entities $\mathbf{C}_1,\dots,\mathbf{C}_m$ have issued bonds and/or stocks which are publicly traded on major financial markets, then typically the default times $\tau_1,\tau_2\dots,\tau_m$ are *directly observable* on the market at the defaults, and these observations are done exogenously without the knowledge of S_t, that is, regardless if the stock price model for S_t includes the defaults or not. Hence, the point process $N_t^{(m)}=\sum_{i=1}^m 1_{\{\tau_i\leq t\}}$ is in practice always observable, making the quantities $\mathbb{E}\left[S_t\mid N_t^{(m)}\right]$ and $\mathbb{E}\left[S_t\mid N_t^{(m)}=k\right]$ given by (2.16) in Theorem 2.14 realistic to compute under the real probability measure $\mathbb{P}$. If the default times $\tau_1,\tau_2\dots,\tau_m$ are *unobservable on the market*, and therefore also making the counting process $N_t^{(m)}$ unobservable, or if the jumps come from a Poisson process with arrival times that lack financial interpretation, and therefore could not be observed directly, then it less clear how to compute, e.g., the quantity $\mathbb{E}\left[S_t\mid N_t^{(m)}\right]$ in practice, since $N_t^{(m)}$ would not be known to us. Note however that $\mathbb{E}[S_t]$ in (2.17) will always be possible to compute, regardless if $N_t^{(m)}$ is observable or not, since to find $\mathbb{E}[S_t]$ we do not need the exact value of $N_t^{(m)}$, but only its distribution.

Note that the η-parameter in the mapping $\Psi_0(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in (2.20) for $k = 0$ will have no impact, and is only present for notational convenience given the sum in the expression of (2.18) which runs from $k = 0$ to $k = m$.

Some remarks on the expected stock price. Let $S_t^{(\mathrm{BS})}$ be the stock price in the Black-Scholes model under the real probability measure $\mathbb{P}$ given by (2.4), that is

$$S_t^{(\mathrm{BS})} = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \tag{2.25}$$

so that

$$\mathbb{E}\left[S_t^{(\mathrm{BS})}\right] = S_0 e^{\mu t}. \tag{2.26}$$

Let S_t be a stock price given by Definition 2.1 under the real probability measure $\mathbb{P}$ and where the jumps $\widetilde{V}_1, \ldots, \widetilde{V}_m$ are distributed as $V_1, \ldots, V_m$ in Definition 2.10 with $\eta > 0$. Then, Equation (2.17) in Theorem 2.14 together with (2.26) implies that

$$\mathbb{E}[S_t] = \mathbb{E}\left[S_t^{(\mathrm{BS})}\right]\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right]. \tag{2.27}$$

We clearly see that if $0 < \eta < \infty$, then $\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right] < 1$, and therefore (2.27) implies the relationship

$$\mathbb{E}[S_t] < \mathbb{E}\left[S_t^{(\mathrm{BS})}\right] \qquad \text{when } 0 < \eta < \infty \tag{2.28}$$

which is intuitively clear since the S_t will always have negative relative jumps at any default time τ_i where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, that is, for the same W_t in S_t given by Corollary (2.13) as in $S_t^{(\mathrm{BS})}$ in (2.25), then Corollary (2.13) implies that $S_t \leq S_t^{(\mathrm{BS})}$ almost surely under $\mathbb{P}$. If $U_n = 0$ for all n (or if "$\eta = \infty$", see in Remark 2.11), this means that there will be no jumps at the defaults τ_i, and S_t will coincide with the Black-Scholes price $S_t^{(\mathrm{BS})}$, that is $S_t = S_t^{(\mathrm{BS})}$ as stated in Equation (2.25), so $\mathbb{E}[S_t] = \mathbb{E}\left[S_t^{(\mathrm{BS})}\right]$. We note that

$$\frac{\partial}{\partial \eta}\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right] = \mathbb{E}\left[\frac{N_t^{(m)}}{\eta^2}\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}+1}\right] > 0 \quad \text{for } \eta > 0 \tag{2.29}$$

so $\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right]$ is strictly increasing in $\eta > 0$. Therefore, for a fixed time point t, and for any $0 < \beta < 1$, the equation $\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right] = \beta$ will have a unique solution in $\eta = \eta(\beta, t)$. This can be used when calibrating η. For example, if we assume that the default counting process $N_t^{(m)}$ will make the expected value of the stock price S_t to $\beta = 90\%$ of the corresponding expected value of the Black stock price $S_t^{(\mathrm{BS})}$, up to time, say T, that is

$$\mathbb{E}[S_T] = \beta\mathbb{E}\left[S_T^{(\mathrm{BS})}\right] \tag{2.30}$$

then (2.27) and (2.30) imply for any $0 < \beta < 1$ that

$$\mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_T^{(m)}}\right] = \beta \tag{2.31}$$

which have a unique solution in $\eta^* = \eta(\beta, T) > 0$, and for most credit portfolio models this solution η^* has to be found numerically. Finally, we will often consider the equation $\mathbb{E}[S_T] = S_0$, so from (2.26), (2.30), and (2.31), we see that when $\mu > 0$, then

$$\mathbb{E}[S_T] = S_0 \quad \text{if and only if} \quad \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_T^{(m)}}\right] = e^{-\mu T} \tag{2.32}$$

where we note that the condition $\mathbb{E}[S_T] = S_0$ implies that the defaults among the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$ "wipes" out the expected log-growth for a corresponding Black-Scholes model with drift μ up to time T. We will use condition (2.32) when calibrating η in our numerical studies presented in Section 7, 8, 9, and 10.

VaR-expressions and related quantities. Given formula (2.21) for the distribution of the stock price loss process $F_{L_t^{(S)}}(x)$ in Theorem 2.14, we will be able to find Value-at-Risk for $L_t^{(S)}$ with confidence level α, denoted by $\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)$, since from (2.14) and the fact that S_t is a continuous random variable, then

$$\mathrm{VaR}_\alpha\left(L_t^{(S)}\right) = F^{-1}_{L_t^{(S)}}(\alpha) \quad \text{so that} \quad F_{L_t^{(S)}}\left(\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)\right) = \alpha \tag{2.33}$$

where the second equation in (2.33) will be solved numerically to find $\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)$. In the case where there are now jump at the defaults, i.e. when $U_n = 0$ for all n, or equivalently, in view of Remark 2.11, if "$\eta = \infty$", then $S_t = S_t^{(\mathrm{BS})}$ with $S_t^{(\mathrm{BS})}$ given by Equation (2.4), and the expression for $\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)$ in (2.33) can then be solved analytically, denoted by $\mathrm{VaR}^{\mathrm{BS}}_\alpha\left(L_t^{(S)}\right)$, and given as

$$\mathrm{VaR}^{\mathrm{BS}}_\alpha\left(L_t^{(S)}\right) = S_0\left(1 - \exp\left(\sigma\sqrt{t}\Phi^{-1}(1-\alpha) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)\right). \tag{2.34}$$

We will later in the numerical section use $\mathrm{VaR}^{\mathrm{BS}}_\alpha\left(L_t^{(S)}\right)$ in (2.34) for the Black-Scholes model when comparing with $\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)$ coming from a stock price S_t with jumps at the default arrivals in $N_t^{(m)}$ and where $\eta > 0$.

We finally remark that almost all formulas in Theorem 2.14 require efficient and quick methods of computing the number of default distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$.

3. **Comparison with stock price models where jump times are driven by Cox processes.** Let S_t be a stock price under the real probability measure $\mathbb{P}$ defined as in Corollary 2.13 where the jump process $N_t^{(m)}$ is given by (2.8), that is $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \le t\}}$ and $\tau_1, \tau_2 \dots, \tau_m$ are the default times of the entities $\mathbf{C}_1, \dots, \mathbf{C}_m$. Here, we remind that the default times $\tau_1, \tau_2 \dots, \tau_m$ creating $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \le t\}}$ can come from any credit portfolio model as long as the log-jumps in the stock price S_t in are independent of these defaults and also independent of the Brownian motion in the exponent of S_t, see also Remark 2.2.

A relevant question is if the model for the stock price S_t in Corollary 2.13 can be obtained, or replicated, by a model where $N_t^{(m)}$ is replaced with a Cox process N_t with some stochastic intensity λ_t adapted to some given filtration $\mathcal{F}_t$. The question can be reformulated as follows: does there exist a Cox process N_t such that $N_t^{(m)}$ is equal in distribution with N_t for the m first jumps for all time points $t > 0$. The family of Cox processes, sometimes also denoted as doubly stochastic Poisson processes, includes, among others, standard Poisson processes, inhomogeneous Poisson processes, Markov-modulated Poisson processes, and many other counting processes. There are several equivalent definitions of a Cox process. Here, we use a somewhat modified version of the definition in [7].

Definition 3.1. Cox processes (doubly stochastic Poisson process). Let N_t be a point process adapted to the filtration $\mathcal{F}_t$, and let λ_t be a nonnegative process such that λ_t is $\mathcal{F}_0$-measurable for all $t \geq 0$ and

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{P}\text{ -a.s.} \quad \text{for all } t \geq 0. \tag{3.1}$$

Then, if for all $0 \leq s \leq t$ and for all integers $k \geq 0$ we have

$$\mathbb{P}\left[N_t - N_s = k \,|\, \mathcal{F}_s \right] = \exp\left(- \int_s^t \lambda_u du \right) \frac{\left(\int_s^t \lambda_u du \right)^k}{k!} \tag{3.2}$$

we say that N_t is a $(\mathbb{P}, \mathcal{F}_t)$-doubly stochastic Poisson process, or for short, a doubly stochastic Poisson process, or with a different terminology, an $\mathcal{F}_t$ Cox process driven by λ_t.

An intuitive way to view a Cox process is that we first draw the realization of λ_t, then conditional of knowing λ_t, we obtain the Cox process as an inhomogeneous Poisson process with intensity λ_t. In practice, the filtration $\mathcal{F}_t$ will often be on the form $\mathcal{F}_t = \mathcal{G}_\infty^X \vee \mathcal{H}_t$ where $\mathcal{G}_t^X = \sigma\left(X_s; s \leq t\right)$ and X_t is a stochastic process, and $\lambda_t = \lambda(X_t)$ for some non-negative mapping $\lambda(\cdot) : \mathbb{R} \mapsto \mathbb{R}^+$. Furthermore, $\mathcal{H}_t$ is typically a filtration generated by N_t or the jump times of N_t up to time t. For more on practical settings regarding filtrations for Cox processes, see in [6].

Remark 3.2. Note that Definition 3.1 for a Cox process with intensity λ_t only makes sense if $\lambda_t > 0$ for some t with strictly positive probability, since if this is not true we have that $\lambda_t = 0$ for all $t \geq 0$ $\mathbb{P}$ a.s, and then (3.2) implies that $N_t = 0$ almost surely under $\mathbb{P}$. Hence, if $\lambda_t = 0$ for all $t \geq 0$ $\mathbb{P}$ a.s, then the Cox process N_t will never jump. Thus, when we talk about a Cox process N_t with intensity λ_t defined as in (3.1), we always assume that $\mathbb{P}\left[\lambda_t > 0 \text{ for some } t\right] > 0$ since otherwise $N_t = 0$ almost surely under $\mathbb{P}$.

If $\{S_k\}$ are the jump times of a Cox process N_t, we obviously have that

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}} \tag{3.3}$$

where $S_k < S_{k+1}$ for $k = 1, 2, \ldots$. One can construct the infinite sequence $\{S_k\}$ recursively via the process λ_t and an infinite i.i.d sequence $\{E_k\}$ of exponentially distributed random variables with parameter one in a similar way as when constructing an inhomogeneous Poisson process with deterministic time dependent intensity.

Now, going back to our original question in this section, we ask if it is possible to "replicate" the stock price model for S_t in Corollary 2.13, either almost surely or in distribution, with a model where $N_t^{(m)}$ is replaced by a Cox process N_t with some stochastic intensity λ_t. Hence, in view of Corollary 2.13, we ask if there exists a Cox process N_t with some stochastic intensity λ_t such that

$$\begin{aligned} &S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{N_t} U_n\right) 1_{\{t<S_{m+1}\}} \\ &\stackrel{d}{=} S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{N_t^{(m)}} U_n\right) \end{aligned} \tag{3.4}$$

for all $t \geq 0$, which is equivalent with asking if there exists a Cox process N_t with some stochastic intensity λ_t such that

$$N_t 1_{\{t<S_{m+1}\}} \stackrel{d}{=} N_t^{(m)} \quad \text{for all } t \geq 0 \tag{3.5}$$

where $N_t^{(m)}$ is defined as in (2.8). Intuitively, it is clear that (3.5) is false since $N_t^{(m)}$ have support on the finite set $\{0, 1, \ldots, m\}$, while N_t have support on the countable infinite set of natural integers as seen in (3.2), and we next formalize this idea as a theorem. Hence, we next prove that (3.5) is impossible for a Cox process N_t with some stochastic intensity λ_t, and therefore (3.4) is impossible for a Cox process N_t.

Theorem 3.3. *Consider an arbitrary credit portfolio model with default times $\tau_1, \tau_2 \ldots, \tau_m$ and the process $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Then, there exists **no** Cox process N_t such that $N_t 1_{\{t<S_{m+1}\}} \stackrel{d}{=} N_t^{(m)}$ for all $t \geq 0$.*

Proof. We prove our theorem by contradiction. First, assume that (3.5) holds, that is, assume that there exists a Cox process N_t with some stochastic intensity λ_t such that

$$N_t 1_{\{t<S_{m+1}\}} \stackrel{d}{=} N_t^{(m)} \quad \text{for all } t \geq 0 \tag{3.6}$$

where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. By Remark 3.2, we have that $\mathbb{P}[\lambda_t > 0 \text{ for some } t] > 0$, since otherwise $N_t = 0$ almost surely under $\mathbb{P}$. Next, for any credit portfolio model with default times $\tau_1, \tau_2 \ldots, \tau_m$, we have

$$\mathbb{P}\left[N_t^{(m)} = k\right] \geq 0 \quad \text{for } k = 0, 1, \ldots, m \text{ and } t \geq 0 \tag{3.7}$$

and

$$\mathbb{P}\left[N_t^{(m)} = k\right] = 0 \quad \text{for } k > m \text{ and } t \geq 0 \tag{3.8}$$

since $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, and can thus never be bigger than m. Hence, it obviously holds that

$$\sum_{k=0}^{m} \mathbb{P}\left[N_t^{(m)} = k\right] = 1 \text{ for } t \geq 0\,. \tag{3.9}$$

Next, since we assume that (3.6) is true, then (3.6) implies that

$$\mathbb{P}\left[N_t 1_{\{t<S_{m+1}\}} = k\right] = \mathbb{P}\left[N_t^{(m)} = k\right] \quad \text{for } k = 0, 1, \ldots, m \text{ and } t \geq 0\,. \tag{3.10}$$

Note that if $N_t 1_{\{t<S_{m+1}\}} = k$ for $k \leq m$, then $S_k \leq t < S_{k+1}$. Also note that $S_k < S_{m+1}$ for $k = 1, \ldots, m$, so if $k \leq m$, then $S_k \leq t < S_{k+1}$ implies that $1_{\{t<S_{m+1}\}} = 1$. Hence, for $k \leq m$, we thus have that

$$\mathbb{P}\left[N_t 1_{\{t<S_{m+1}\}} = k\right] = \mathbb{P}\left[N_t = k\right] \quad \text{if } k \leq m \text{ and for any } t \geq 0\,. \tag{3.11}$$

So, if (3.6) is true, we combine (3.10)-(3.11), and then get

$$\mathbb{P}\left[N_t = k\right] = \mathbb{P}\left[N_t^{(m)} = k\right] \quad \text{for } k = 0, 1, \ldots, m \text{ and } t \geq 0\,. \tag{3.12}$$

Hence, if (3.6) is true, then (3.9) and (3.12) imply that

$$1 = \sum_{k=0}^{m} \mathbb{P}\left[N_t^{(m)} = k\right] = \sum_{k=0}^{m} \mathbb{P}\left[N_t = k\right] \text{ for } t \geq 0\,. \tag{3.13}$$

On the other hand, letting $s = 0$ in (3.2) and taking the expected value, we get that

$$\mathbb{P}\left[N_t = k\right] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_u du\right)\frac{\left(\int_0^t \lambda_u du\right)^k}{k!}\right] > 0 \quad \text{for } k \geq 0 \text{ and } t \geq 0 \tag{3.14}$$

where the strictly positive probabilities in (3.14) follow from Remark 3.2. Furthermore, by using the Taylor-expansion of e^x together with standard rules for expectations and (3.14), it is easy to see that for all $t \geq 0$, it holds that $\sum_{k=0}^{\infty} \mathbb{P}\left[N_t = k\right] = 1$ since

$$\sum_{k=0}^{\infty} \mathbb{P}\left[N_t = k\right] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_u du\right)\sum_{k=0}^{\infty}\frac{\left(\int_0^t \lambda_u du\right)^k}{k!}\right] = 1 \quad \text{for } t \geq 0\,. \tag{3.15}$$

Hence, we have that $1 = \sum_{k=0}^{\infty} \mathbb{P}\left[N_t = k\right]$ for $t \geq 0$, so (3.13) and (3.15) then imply that

$$\mathbb{P}\left[N_t = k\right] = 0 \quad \text{for } k > m \text{ and } t \geq 0 \tag{3.16}$$

which contradicts (3.14) for $k > m$ with non-zero intensity λ_t with positive probability; see also Remark 3.2. Hence, the assumption in (3.6) is therefore false, that is, assumption (3.5) is false, and we can thus never find a Cox process N_t with non-zero intensity λ_t with positive probability such that $N_t 1_{\{t < S_{m+1}\}}$ is equal in distribution with $N_t^{(m)}$ for all $t \geq 0$. This concludes the theorem. $\square$

Next, assume that there are no joint defaults among the default times $\{\tau_i\} = \{\tau_1, \tau_2 \ldots, \tau_m\}$, that is, for $i \neq j$, assume that $\mathbb{P}\left[\tau_i = \tau_j\right] = 0$. Let $\{T_k\}$ be the ordering of the default times $\{\tau_i\}$, and since there are no joint defaults, then T_k are well defined for $k = 1, 2, \ldots, m$ and $T_1 < T_2 < \ldots < T_m$.

Recall that the infinite sequence $\{S_k\} = \{S_1, S_2, \ldots\}$ are the jump times of the Cox process so that (3.3) holds, that is, $N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}$. In view of the above notation together with the assumption of no joint defaults among $\{\tau_i\}$, we can now state the following corollary to Theorem 3.3.

Corollary 3.4. *Consider an arbitrary credit portfolio model with default times $\tau_1, \tau_2 \ldots, \tau_m$ with no joint defaults, and let $\{T_k\}$ be the ordering of $\{\tau_i\}$. Then, T_k can never have the same distribution as S_k for all $k \leq m$, where $\{S_k\}$ are the jump times of a Cox process.*

Proof. Let N_t be a Cox process so that (3.3) then implies

$$S_k = \inf\left\{t > 0 : N_t \geq k\right\} \quad \text{for } k = 1, 2, 3, \ldots$$

and thus

$$\mathbb{P}\left[S_k \leq t\right] = \mathbb{P}\left[N_t \geq k\right] = 1 - \mathbb{P}\left[N_t \leq k-1\right] \quad \text{for } t > 0 \text{ and } k = 1, 2, 3, \ldots \tag{3.17}$$

For $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, we similarly get

$$T_k = \inf\left\{t > 0 : N_t^{(m)} \geq k\right\} \qquad \text{for } k = 1, 2, \ldots, m$$

so that

$$\mathbb{P}\left[T_k \leq t\right] = \mathbb{P}\left[N_t^{(m)} \geq k\right] = 1 - \mathbb{P}\left[N_t^{(m)} \leq k-1\right] \qquad \text{for } t > 0 \text{ and } k = 1, 2, \ldots, m\,. \tag{3.18}$$

From Theorem 3.3, we know that there is no Cox process N_t with non-zero intensity λ_t with positive probability such that $N_t 1_{\{t < S_{m+1}\}}$ is equal in distribution with $N_t^{(m)}$ for all $t \geq 0$. Thus, from (3.17) and (3.18) it is clear that S_k can never be equal in distribution with T_k for $k = 1, 2, \ldots, m$, and this concludes the corollary. □

There exist several stock price models with jumps that builds on the model by [32], but where the Poisson process in [32] is replaced with some Cox process. For example, in [9], the authors build on [32], but use a Markov-modulated Poisson process instead of a Poisson process. A Markov-modulated Poisson process is a Cox process where the stochastic intensity λ_t is driven by a finite state continuous time Markov Chain.

Remark 3.5. The stock price model in this paper can never be replicated by an extension of [32] with some Cox process. In view of Theorem 3.3 and Corollary 3.4, we make the following observations. A stock price model with jumps at the default times $\tau_1, \tau_2 \ldots, \tau_m$, as in Corollary 2.13, can never be equal in distribution with a model where the default times are replaced with the m first jumps of some Cox process, in particular not a Markov-modulated Poisson process. Hence, the model presented in this paper is unique in the sense that it can not be seen as a special case of the paper [9], [32], or any other model based on [32] where the Poisson process is replaced with a Cox process and where all jumps in the stock price are negative and have the same distribution. Furthermore, since our model does not put any restriction on the type of default model for $\tau_1, \tau_2 \ldots, \tau_m$, the distribution of S_t can vary a lot depending on the choice of credit portfolio model for $\tau_1, \tau_2 \ldots, \tau_m$. In the numerical sections, we consider three different models: A model with CIR-intenisites for the default times, a one-factor Gaussian copula model for τ_i, and a Clayton copula model for the default times τ_i. Other credit models that can be chosen are, for example, default contagion models such as those in [11, 17, 22, 23, 27, 34], and such default contagion models produce much fatter tails for the process $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ compared with Gaussian copula models, CIR-intensity models, or other conditional independent default intensity models.

Even though Theorem 3.3 and Corollary 3.4 show that it is impossible to construct the stock price model in Corollary 2.13 by using a Cox process N_t instead of the default point process $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, it is still interesting to compare the stock price model in Corollary 2.13 where $N_t^{(m)}$ is replaced with a Poisson process. Such a comparison will be done in Section 6 producing a version of the Kou model, [32], with only negative jumps, and then in Section 10 we numerically compare our stock price model from Corollary 2.13 with the Kou model containing only negative jumps as outlined in Section 6.

4. The multidimensional case: Small time approximations to loss distributions for heterogeneous stock portfolios with jumps at exogenous

defaults. In this section, we generalize the single-stock dynamics in Section 2 to a heterogeneous portfolio of stocks. Furthermore, we also define the loss process for the stock portfolio. For small time points we make a linearization of the portfolio loss process and derive a computationally tractable expression for distribution of the linearized loss. We also consider the portfolio loss process and its linear approximation for small time points in the classical Black-Scholes portfolio case, i.e. without any jumps in the stock prices. In our numerical studies in Section 8, we will use the distribution of the linearized Black-Scholes portfolio loss as a benchmark for the distribution of the linearized loss when the stock prices have jumps at defaults of some external defaultabe entities.

Inspired by the dynamics of a single-stock price S_t discussed in Section 2, and in particular Corollary 2.13, we now give the following definition.

Definition 4.1. Consider a group of m defaultable entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ with individual default times $\tau_1, \tau_2 \ldots, \tau_m$, and let $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$. Let companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ be J different exchangeable entities which do not belong to the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$, and let $S_{t,1}, \ldots, S_{t,J}$ denote the stock prices of companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ at time t under the real probability measure $\mathbb{P}$. Then, for each entity $\mathbf{A}_j$, we define the stock price $S_{t,j}$ as

$$S_{t,j} = S_{0,j} \exp\left(\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t + \sigma_j\left(\rho_{S,j}W_{t,0} + \sqrt{1-\rho_{S,j}^2}W_{t,j}\right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j}\right) \tag{4.1}$$

where $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ are $J+1$ independent Brownian motions, and $\rho_{S,j} \in [-1,1]$ are constants. Furthermore, for each $j = 1, 2, \ldots, J$, the m random variables $U_{1,j}, \ldots, U_{m,j}$ are an i.i.d sequence distributed as

$$U_{n,j} \stackrel{d}{=} \text{Exp}(\eta) \quad \text{with} \quad \mathbb{E}\left[U_{n,j}\right] = \frac{1}{\eta} \tag{4.2}$$

where $U_{1,j}, \ldots, U_{m,j}$ are independent of the processes $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ and also independent of the default times $\tau_1, \tau_2 \ldots, \tau_m$. Furthermore, for each company $\mathbf{A}_j$, the parameters $\sigma_j > 0$ and μ_j are the volatility and drift, the same as in the one-dimensional case given in Definition 2.1 and Corollary 2.13.

Next, we make some remarks connected to Definition 4.1.

Remark 4.2. If we let $\tilde{U}_{1,j}, \ldots, \tilde{U}_{m,j}$ be an i.i.d sequence with the same distribution as $U_{1,j}, \ldots, U_{m,j}$, then the jump term $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in (4.1) can be replaced by the more intuitive expression $\sum_{i=1}^m \tilde{U}_{i,j} 1_{\{\tau_i \leq t\}}$, just as in the single-stock case in Section 2, since $\sum_{n=1}^{N_t^{(m)}} U_{n,j} \stackrel{d}{=} \sum_{i=1}^m \tilde{U}_{i,j} 1_{\{\tau_i \leq t\}}$. However, in the derivations in our proofs, it will be more convenient from a notational point of view to use the first version, that is, the term $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in (4.1).

Remark 4.3. Note that in Definition 4.1, all firms $\mathbf{A}_j$ have stock prices $S_{t,j}$ with i.i.d jumps $U_{1,j}, \ldots, U_{m,j}$, with the parameter η defined as in (4.2). We can of course also let the distributions for $U_{1,j}, \ldots, U_{m,j}$ be different among different entities $\mathbf{A}_j$, for example, by letting

$$U_{n,j} \stackrel{d}{=} \text{Exp}(\eta_j) \quad \text{with} \quad \mathbb{E}\left[U_{n,j}\right] = \frac{1}{\eta_j} \quad \text{where } \eta_j \neq \eta_i \quad \text{for } \mathbf{A}_j \neq \mathbf{A}_i\,. \tag{4.3}$$

However, allowing for heterogeneous jump parameters η_j among different firms $\mathbf{A}_j$, as in (4.3), will unfortunately make it difficult to find analytical formulas for the distribution of our stock portfolio losses. Therefore, in this paper we will always assume homogeneous jump parameters, that is, $\eta = \eta_1 = \eta_2 = \ldots \eta_J$, which will lead to analytical formulas for our portfolio related quantities.

Remark 4.4. Note that $\rho_{S,j} \in [-1,1]$, and, unless explicitly stated, throughout this paper we will always assume that at least one company $\mathbf{A}_j$ has a correlation such that $\rho_{S,j} \neq -1, 1$ so that $\rho_{S,j} \in (-1,1)$.

Remark 4.5. Since the collection of i.i.d sequences $U_{1,j}, \ldots, U_{m,j}$ are exchangeable for all companies $\mathbf{A}_j$ in Definition 4.1, that is $U_{k,j}$ and $U_{k',j'}$ have the same distribution for any pairs (k,j) and (k',j'), then, just as in Remark 2.2, we note that the default times $\tau_1, \tau_2 \ldots, \tau_m$ in Definition 4.1 can come from any type credit portfolio model.

Remark 4.6. The stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ are correlated and have simultaneous jumps. Since $W_{t,0}$ and $W_{t,j}$ are independent Brownian motions for each j and $\rho_{S,j} \in [-1,1]$, then from standard probability theory we know that $\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2} W_{t,j}$ used in (4.1) is also a Brownian motion. Hence, in view of Definition 2.1, Definition 2.10, and Corollary 2.13, it is clear that the dynamics of the stock price $S_{t,j}$ for each firm $\mathbf{A}_j$ satisfies

$$dS_{t,j} = S_{t-,j} dY_{t,j} \tag{4.4}$$

where $Y_{t,j}$ is given by

$$Y_{t,j} = \mu_j t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2} W_{t,j} \right) + \sum_{n=1}^{N_t^{(m)}} \left(e^{-U_{n,j}} - 1 \right) . \tag{4.5}$$

Further, from the construction of $S_{t,j}$ in (4.1) and $U_{n,j}$ in (4.2), stated in Definition 4.1, the stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ will be "correlated" via the factor process $W_{t,0}$ when $\rho_{S,j} \neq 0$, and also "correlated" via the default counting process $N_t^{(m)}$ for the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$. In particular, all stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ will have a jump at the default times $\tau_1, \tau_2 \ldots, \tau_m$, where the relative jumps of $S_{t,j}$ will be different almost surely under $\mathbb{P}$, although they have same distribution given by (4.2). Finally, each stock price $S_{t,j}$ will satisfy the results in Theorem 2.14.

Next, consider a weighted stock portfolio consisting of $w_1, w_2, \ldots, w_J$ stocks chosen for our portfolio at time $t = 0$, where the stocks are issued by the J companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ with stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ that satisfy Definition 4.1. Then, we define the portfolio value V_t as

$$V_t = \sum_{j=1}^{J} w_j S_{t,j} . \tag{4.6}$$

We will in this paper define an equally value-weighted portfolio V_t as follows.

Definition 4.7. Equally value-weighted portfolio. Let S_0 be a positive constant. We say that the portfolio V_t in (4.6) is an *equally value-weighted portfolio* if the weights w_j are chosen so that

$$w_j S_{0,j} = S_0 \quad \text{for} \quad j = 1, 2, \ldots, J \tag{4.7}$$

and thus

$$V_0 = \sum_{j=1}^{J} w_j S_{0,j} = \sum_{j=1}^{J} S_0 = J S_0 \,. \tag{4.8}$$

The intuitive idea behind Definition 4.7 is that the portfolio weights w_j are chosen so that the value for the stock position in firm $\mathbf{A}_j$ at time $t=0$ will have the same amount given by S_0 for all companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ that are contained in the portfolio V_t.

Next, we define the portfolio loss process $L_t^{(V)}$ for a general portfolio V_t at time t with reference to the starting time 0 as

$$L_t^{(V)} = -\left(V_t - V_0\right) \tag{4.9}$$

where we note that a gain implies that the loss $L_t^{(V)}$ is negative. We are interested in computing Value-at-Risk for $L_t^{(V)}$ in our model given by Definition 4.1, that is, we want to compute

$$\mathrm{VaR}_\alpha\left(L_t^{(V)}\right) = \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[L_t^{(V)} > y\right] \leq 1-\alpha\right\} = \inf\left\{y \in \mathbb{R} : F_{L_t^{(V)}}(y) \geq \alpha\right\} \tag{4.10}$$

where $F_{L_t^{(V)}}(x)$ is the distribution of $L_t^{(V)}$ and α is the confidence level, just as in (2.14). Define $X_{t,j}$ as

$$X_{t,j} = \left(\mu_j - \frac{1}{2}\sigma_j^2\right) t + \sigma_j\left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2}\, W_{t,j}\right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \tag{4.11}$$

where the right hand side of (4.11) is the same as in (4.1) in Definition 4.1, which then implies that

$$S_{t,j} = S_{0,j} e^{X_{t,j}} \,. \tag{4.12}$$

Then, for an equally value-weighted portfolio V_t as in Definition 4.7, the portfolio loss $L_t^{(V)}$ in (4.9) can be restated as

$$L_t^{(V)} = S_0\left(J - \sum_{j=1}^{J} e^{X_{t,j}}\right) \,. \tag{4.13}$$

We want to find $F_{L_t^{(V)}}(x) = \mathbb{P}\left[L_t^{(V)} \leq x\right]$ so that we, for example, can compute $\mathrm{VaR}_\alpha\left(L_t^{(V)}\right)$ given by (4.10). Unfortunately, finding analytical or semi-analytical expressions of $F_{L_t^{(V)}}(x)$ is a challenging task. However, assuming that $|X_{t,j}|$ will be small for small t, we can use a first-order Taylor expansion of the term $e^{X_{t,j}}$, that is

$$e^{X_{t,j}} \approx 1 + X_{t,j} \quad \text{when } |X_{t,j}| \text{ is small} \tag{4.14}$$

which typically will hold for small t. So, using (4.14) in (4.13) then implies that the loss $L_t^{(V)}$ for an equally value-weighted portfolio V_t as in Definition 4.7 is approximated by

$$L_t^{(V)} \approx -S_0 \sum_{j=1}^{J} X_{t,j} \quad \text{when } |X_{t,j}| \text{ is small for all } j \,. \tag{4.15}$$

For $X_{t,j}$ given by (4.11), we therefore define the linearized loss $L_t^{\Delta V}$ to the portfolio loss $L_t^{(V)}$ in an equally value-weighted portfolio as

$$L_t^{\Delta V} = -S_0 \sum_{j=1}^{J} X_{t,j} \tag{4.16}$$

so that (4.15) then implies that

$$\mathbb{P}\left[L_t^{(V)} \leq x\right] \approx \mathbb{P}\left[L_t^{\Delta V} \leq x\right] \quad \text{when } |X_{t,j}| \text{ is small for all } j \tag{4.17}$$

which typically will hold for small t. Next, we state a theorem which provides a computationally tractable semi-analytical expression to the distribution $\mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ for the linearized loss $L_t^{\Delta V}$ defined as in (4.16), which is equivalent to finding the distribution of $\sum_{j=1}^{J} X_{t,j}$.

Theorem 4.8. *Consider an equally value-weighted portfolio as in Definition 4.7 where the J stock prices $S_{t,1}, \ldots, S_{t,J}$ are defined as in Definition 4.1 under the real probability measure $\mathbb{P}$. Then, with notation as above, we have that*

$$\mathbb{P}\left[L_t^{\Delta V} \leq x\right] = 1 - \sum_{k=0}^{m} \Psi_k^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right) \mathbb{P}\left[N_t^{(m)} = k\right] \tag{4.18}$$

where the mappings $\Psi_k^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ for $k \geq 1$ are defined as

$$\begin{aligned}&\Psi_k^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)\\ &= \int_0^\infty \Phi\left(\frac{y - \frac{x}{S_0} - \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t}{\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}}\right)\frac{\eta e^{-\eta y}\left(\eta y\right)^{Jk-1}}{(Jk-1)!}\,dy\end{aligned} \tag{4.19}$$

and for $k = 0$, the mapping $\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ is defined by

$$\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right) = \Phi\left(\frac{-\frac{x}{S_0} - \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t}{\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}}\right) \tag{4.20}$$

where $\Phi(x)$ and $\varphi(x)$ are the distribution function and density to a standard normal random variable.

Proof. First, since $S_0 > 0$, and in view of (4.16), after some rearranging, we get

$$\mathbb{P}\left[L_t^{\Delta V} \leq x\right] = 1 - \mathbb{P}\left[\sum_{j=1}^{J} X_{t,j} \leq -\frac{x}{S_0}\right] \tag{4.21}$$

and we therefore seek the distribution of $\sum_{j=1}^{J} X_{t,j}$. From Definition 4.1 and (4.11), we can rewrite $X_{t,j}$ as

$$X_{t,j} = Z_{t,j} + \left(\mu_j - \frac{1}{2}\sigma_j^2\right)t - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \tag{4.22}$$

where $Z_{t,j}$ is defined by

$$Z_{t,j} = \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2} W_{t,j} \right) \tag{4.23}$$

and the terms on the right-hand side of (4.23) are the same as in Equation (4.1) in Definition 4.1. Then,

$$\mathbb{P}\left[\sum_{j=1}^{J} X_{t,j} \leq -\frac{x}{S_0}\right] = \mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq -\frac{x}{S_0} - \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right) t\right] \tag{4.24}$$

For notational convenience, we define $a(x)$ as

$$a(x) = -\frac{x}{S_0} - \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right) t \tag{4.25}$$

so that (4.24) can be rewritten as

$$\mathbb{P}\left[\sum_{j=1}^{J} X_{t,j} \leq -\frac{x}{S_0}\right] = \mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq a(x)\right]. \tag{4.26}$$

Next, we note that

$$\begin{aligned} &\mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{N_t^{(m)}} U_{n,j} \leq a(x)\right] \\ &= \sum_{k=0}^{m} \mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{k} U_{n,j} \leq a(x) \,\middle|\, N_t^{(m)} = k\right] \mathbb{P}\left[N_t^{(m)} = k\right] \end{aligned} \tag{4.27}$$

and since $W_{t,j}$ and $U_{n,j}$ are independent of $N_t^{(m)}$ for all j and n, then by using the same arguments which led to the right-hand side in (A.2.3) in Theorem 2.14, we get

$$\begin{aligned} &\mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{k} U_{n,j} \leq a(x) \,\middle|\, N_t^{(m)} = k\right] \\ &= \mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{k} U_{n,j} \leq a(x)\right]. \end{aligned} \tag{4.28}$$

From the definition of $Z_{t,j}$ in (4.23), we have that

$$\begin{aligned} \sum_{j=1}^{J} Z_{t,j} &= \sum_{j=1}^{J} \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2} W_{t,j} \right) \\ &= W_{t,0} \sum_{j=1}^{J} \sigma_j \rho_{S,j} + \sum_{j=1}^{J} \sigma_j \sqrt{1-\rho_{S,j}^2} W_{t,j} \end{aligned} \tag{4.29}$$

and since $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ are $J+1$ independent Brownian motions, then (4.29) and standard results from probability theory together with some computations give that

$$\sum_{j=1}^{J} Z_{t,j} \stackrel{d}{=} \sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2+\sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}X \tag{4.30}$$

where X is a standard normal random variable. Let G_{Jk} be random variables independent of X where G_{Jk} is a gamma-distributed random variable such that $G_{Jk} \stackrel{d}{=} \text{Gamma}(Jk,\eta)$ where $k \geq 1$ is an integer. Then, in view of Definition 4.1, standard probability theory, and using the same arguments that led to (A.2.5) in Theorem 2.14, we have that

$$\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{k} U_{n,j} \stackrel{d}{=} \sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2+\sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}X - G_{Jk}\,. \tag{4.31}$$

Next, by using (4.31) in a version of Equation (A.2.11) in Theorem 2.14, we obtain

$$\begin{aligned}&\mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} - \sum_{j=1}^{J}\sum_{n=1}^{k} U_{n,j} \leq a(x)\right]\\ &= \int_0^\infty \Phi\left(\frac{a(x)+y}{\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2+\sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}}\right) f_{G_{Jk}}(y)dy \end{aligned}\tag{4.32}$$

where $f_{G_{Jk}}(y) = \frac{\eta e^{-\eta y}(\eta y)^{Jk-1}}{(Jk-1)!}$ is the density of G_{Jk} and $\Phi(x)$ is the distribution function to a standard normal random variable. If $k = 0$, there are no jump-terms, so the right-hand side of (4.28) reduces to

$$\mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} \leq a(x)\right] = \Phi\left(\frac{a(x)}{\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2+\sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}}\right) \tag{4.33}$$

where we also used (4.30) for the distribution of $\sum_{j=1}^{J} Z_{t,j}$. Hence, using (4.32) for $k \geq 1$ and (4.33) for $k = 0$ on the right-hand side of (4.28) and (4.27) and (4.26) together with the definition of $a(x)$ in (4.25) finally implies that (4.21) can be rewritten as

$$\mathbb{P}\left[L_t^{\Delta V} \leq x\right] = 1 - \sum_{k=0}^{m} \Psi_k^V\left(x,t,\mu,\sigma,S_0,\rho_S,\eta\right)\mathbb{P}\left[N_t^{(m)} = k\right]$$

where the mappings $\Psi_k^V\left(x,t,\mu,\sigma,S_0,\rho_S,\eta\right)$ for $k > 1$ are defined by

$$\Psi_k^V\left(x,t,\mu,\sigma,S_0,\rho_S,\eta\right)$$

$$= \int_0^\infty \Phi\left(\frac{y - \frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right) t}{\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2 \left(1 - \rho_{S,j}^2\right)\right)}}\right) \frac{\eta e^{-\eta y} \left(\eta y\right)^{Jk-1}}{(Jk-1)!} dy$$

and for $k = 0$ the mapping $\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ is defined as

$$\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right) = \Phi\left(\frac{- \frac{x}{S_0} - \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right) t}{\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2 \left(1 - \rho_{S,j}^2\right)\right)}}\right)$$

proving (4.18), (4.19), and (4.20), which concludes the theorem. □

We note that the η-parameter in the mapping $\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ in (4.20) for $k = 0$ will have no impact, and is only present for notational convenience given the sum in the expression of (4.18) which runs from $k = 0$ to $k = m$.

Remark 4.9. Note that Theorem 4.8 is stated for a heterogeneous stock portfolio so that the parameters $\mu_j, \sigma_j, \rho_{S,j}$, and $S_{0,j}$ can have different values for different firms $\mathbf{A}_j$, but where the weights w_j in the portfolio V_t are chosen so that $w_j S_{0,j} = S_0$ for all companies where S_0 is a positive constant. Sometimes, we want to study the case where the parameters for $S_{t,j}$ are identical for all firms $\mathbf{A}_j$, that is, when

$$S_{0,j} = S_0, \quad \mu_j = \mu, \quad \sigma_j = \sigma, \quad \text{and} \quad \rho_{S,j} = \rho_S \quad \text{for all firms} \quad \mathbf{A}_1, \ldots, \mathbf{A}_J \tag{4.34}$$

so that the stock prices $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ become exchangeable. Furthermore, by letting $w_j = 1$ for all companies, we get an equally value-weighted portfolio as in Definition 4.7, and (4.34) together with Theorem 4.8 then imply that the mappings $\Psi_k^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ in the loss distribution $\mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ given by (4.18) will simplify a bit, where for $k \geq 1$ under (4.34), we get

$$\Psi_k^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right) = \int_0^\infty \Phi\left(\frac{y - \frac{x}{S_0} - J\left(\mu - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{tJ\left(1 + (J-1)\rho_S^2\right)}}\right) \frac{\eta e^{-\eta y} \left(\eta y\right)^{Jk-1}}{(Jk-1)!} dy \tag{4.35}$$

and for $k = 0$ with condition (4.34), the mapping $\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right)$ is simplified to

$$\Psi_0^V\left(x, t, \mu, \sigma, S_0, \rho_S, \eta\right) = \Phi\left(\frac{- \frac{x}{S_0} - J\left(\mu - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{tJ\left(1 + (J-1)\rho_S^2\right)}}\right) \tag{4.36}$$

where the rest of the notation is same as in Theorem 4.8.

Given formulas (4.18)-(4.20) in Theorem 4.8 for the distribution $F_{L_t^{\Delta V}}(x) = \mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ where $L_t^{\Delta V}$ is the linear approximation to the portfolio loss $L_t^{(V)}$, we can find the Value-at-Risk for $L_t^{\Delta V}$ with confidence level α, denoted by $\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right)$, as

$$\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right) = F_{L_t^{\Delta V}}^{-1}(\alpha) \quad \text{so that} \quad F_{L_t^{\Delta V}}\left(\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right)\right) = \alpha \tag{4.37}$$

since $L_t^{\Delta V}$ is a continuous random variable. Equation (4.37) can for most credit portfolio models only be solved numerically. Also, note that $\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right)$ will for small time points t be an approximation to $\mathrm{VaR}_\alpha\left(L_t^{(V)}\right)$ as defined in (4.10), that is

$$\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right) \approx \mathrm{VaR}_\alpha\left(L_t^{(V)}\right) \quad \text{so} \quad F^{-1}_{L_t^{\Delta V}}(\alpha) \approx F^{-1}_{L_t^{(V)}}(\alpha) \quad \text{for small time points } t. \tag{4.38}$$

Just as in Theorem 2.14, we again remark that the formulas in Theorem 4.8 and related computations as in (4.37) require efficient and quick methods of computing the number of the default distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$.

In our numerical studies in Sections 7 - 8, we will use the results in Theorem 4.8 together with efficient numerical methods for computing $\mathbb{P}\left[N_t^{(m)} = k\right]$ in an intensity-based CIR model, and also in a one-factor Gaussian copula model.

Remark 4.10. In the case when there is no jump at the defaults in Definition 4.1, i.e. when $U_n = 0$ for all n, then $S_{t,j} = S_{t,j}^{(\mathrm{BS})}$ for all companies $\mathbf{A}_j$ with $S_{t,j}^{(\mathrm{BS})}$ given by

$$S_{t,j}^{(\mathrm{BS})} = S_{0,j} \exp\left(\left(\mu_j - \frac{1}{2}\sigma_j^2\right) t + \sigma_j \left(\rho_{S,j} W_{t,0} + \sqrt{1-\rho_{S,j}^2}\, W_{t,j}\right)\right) \tag{4.39}$$

where $W_{t,0}, W_{t,1}, \ldots, W_{t,J}$ are $J+1$ independent Brownian motions, and the rest of the notation is the same as in Definition 4.1. Note that $S_{t,1}^{(\mathrm{BS})}, \ldots, S_{t,J}^{(\mathrm{BS})}$ under (4.39) will still be correlated via the factor process $W_{t,0}$, and recall that $\rho_S W_{t,0} + \sqrt{1-\rho_S^2}\, W_{t,j}$ is a Brownian motion for each stock price $S_{t,j}^{(\mathrm{BS})}$.

In view of Remark 4.10, we now state the following corollary to Theorem 4.8 in the case where there are no jumps among the stock prices $S_{t,j}$.

Corollary 4.11. *Consider an equally value-weighted portfolio as in Definition 4.7 where the J stock prices $S_{t,1}^{(BS)}, \ldots, S_{t,J}^{(BS)}$ are defined as in (4.39) under the real probability measure $\mathbb{P}$. Then, with notation as above,*

$$\mathbb{P}\left[L_t^{\Delta V} \le x\right] = \Phi\left(\frac{\frac{x}{S_0} + \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right) t}{\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2 \left(1-\rho_{S,j}^2\right)\right)}}\right) \tag{4.40}$$

and

$$\begin{aligned} &VaR_\alpha\left(L_t^{\Delta V}\right) \\ &= S_0\left(\sqrt{t\left(\left(\sum_{j=1}^J \sigma_j \rho_{S,j}\right)^2 + \sum_{j=1}^J \sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}\,\Phi^{-1}(\alpha) - \sum_{j=1}^J \left(\mu_j - \frac{1}{2}\sigma_j^2\right) t\right) \end{aligned} \tag{4.41}$$

where $\Phi(x)$ is the distribution function to a standard normal random variable. Furthermore, if the stock prices $S_{t,1}^{(BS)}, \ldots, S_{t,J}^{(BS)}$ also satisfy (4.34) in Remark 4.9, then (4.40)-(4.41) simplify to

$$\mathbb{P}\left[L_t^{\Delta V} \le x\right] = \Phi\left(\frac{\frac{x}{S_0} + J\left(\mu - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{tJ\left(1+(J-1)\rho_S^2\right)}}\right) \tag{4.42}$$

and

$$VaR_{\alpha}\left(L_t^{\Delta V}\right) = S_0\left(\sigma\sqrt{tJ\left(1+(J-1)\rho_S^2\right)}\Phi^{-1}(\alpha) - J\left(\mu - \frac{1}{2}\sigma^2\right)t\right). \quad (4.43)$$

A proof of Corollary 4.11 is given in Subsection A.3 of Appendix A.

In our numerical studies in Section 7 and 8, we will use the "Black-Scholes" linear portfolio formulas in Corollary 4.11 as a benchmark for expressions of the stock prices with jumps at defaults given in Theorem 4.8.

The results in Theorem 4.8 and Corollary 4.11 hold for heterogeneous stock portfolios which are equally value-weighted and have arbitrary size J, that is, the number of stocks J in the portfolio can be small or large. The main drawback with the formulas in Theorem 4.8 and Corollary 4.11 is that these expressions for the linearized loss $L_t^{\Delta V}$ only work somewhat accurately as an approximation of the true loss $L_t^{(V)}$ when the time t is small, and the expressions will fail as time t starts to increase. For example, the linearized loss $L_t^{\Delta V}$ may produce VaR-values that are bigger than V_0, which is impossible since, by construction, it will hold that $L_t^{(V)} \leq V_0$ almost surely for all $t \geq 0$ under the real probability measure $\mathbb{P}$. However, in certain cases we can still find highly analytical approximation formulas for the loss distribution $\mathbb{P}\left[L_t^{(V)} \leq x\right]$ at any time point t and where the loss will never exceed V_0, as will be seen in the next section.

5. **The multidimensional case: Approximation formulas to loss distributions for large homogeneous stock portfolios with jumps at exogenous defaults.** For larger time points t, the linear approximations to the stock portfolio in Theorem 4.8 and Corollary 4.11 will fail. If we however assume that the stock prices $S_{t,j}$ satisfy (4.34) in Remark 4.9, that is, $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$, and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$ so that the $S_{t,1}, S_{t,2}, \ldots, S_{t,J}$ are exchangeable and the portfolio becomes homogeneous (given same weights), and if the number of stocks J in the portfolio are "large", then we will in this section derive approximation formulas for the loss distribution $\mathbb{P}\left[L_t^{(V)} \leq x\right]$, which will work for arbitrary time points t, that is, both for large and small time points t, and which will also guarantee that portfolio loss will always be smaller than V_0 almost surely for all $t \geq 0$ under the measure $\mathbb{P}$. Hence, in this section we will make two assumptions. First, we assume that (4.34) holds together with Definition 4.1 under the real probability measure $\mathbb{P}$, with equal portfolio weights w_j for all companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ in the portfolio V_t. Our second assumption is that the number of stocks J in the portfolio are "large". Since the stock portfolio is equally weighted and we are primarily interested in Value-at-Risk calculation of the portfolio, then due to the linearity of VaR we can without loss of generality let $w_j = 1$ for each stock in the portfolio, and thus define the portfolio value as $V_t = \sum_{j=1}^J S_{t,j}$. Due to condition (4.34), the portfolio V_t will then be an equally value-weighted portfolio as in Definition 4.7.

Remark 5.1. Homogenization of a heterogeneous stock portfolio: Assuming a completely homogeneous stock portfolio so that the parameters for each stock are the same is of course an unrealistic feature. Consider a heterogeneous stock portfolio with stocks defined as in Definition 4.1, portfolio value $\hat{V}_t$, and define S_0, μ, σ, and ρ_S as the corresponding sample means of the parameters in this portfolio, that is,

$$S_0=\frac{1}{J}\sum_{j=1}^{J}S_{0,j}\qquad \mu=\frac{1}{J}\sum_{j=1}^{J}\mu_j\qquad \sigma=\frac{1}{J}\sum_{j=1}^{J}\sigma_j\qquad\text{and}\qquad \rho_S=\frac{1}{J}\sum_{j=1}^{J}\rho_{S,j}\ . \tag{5.1}$$

Next, create a homogeneous stock portfolio as in Section 4 with parameters S_0,μ,σ, and ρ_S as in (5.1) with portfolio value V_t, and where $W_{t,0},W_{t,1},\dots,W_{t,J}$, $N_t^{(m)}$, and $U_{i,j}$ are the same as in the heterogeneous portfolio. For such homogeneous portfolios, [30] as well as [38] proved that the value process $\hat{V}_t$ for a large heterogeneous stock portfolio can be approximated arbitrarily well by V_t in the L_1-sense as $J\to\infty$. [30] proved the result for portfolios with only diffusions while [38] extended the proof to the case where the stocks can also jump due to Poisson processes. In view of the results of [30] and [38], it is therefore still relevant to consider homogeneous stock portfolios in particular if these portfolios come from doing a homogenization of a heterogeneous stock portfolio as in (5.1).

Given the assumption that (4.34) is satisfied, we now state the following theorem.

Theorem 5.2. *Let $S_{t,1},\dots,S_{t,J}$ be stock prices defined as in Definition 4.1, which satisfies (4.34) under the real probability measure $\mathbb{P}$. Then, with notation as above, we have*

$$\lim_{J\to\infty}\frac{1}{J}\sum_{j=1}^{J}S_{t,j}=S_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \tag{5.2}$$

almost surely under the probability measure $\mathbb{P}\left[\cdot\,|\,W_{t,0},N_t^{(m)}\right]$ and

$$\begin{aligned}&\lim_{J\to\infty}\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J}S_{t,j}\le x\right]\\&=\mathbb{P}\left[S_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\le x\right]\ .\end{aligned} \tag{5.3}$$

Furthermore, for large J we have

$$\begin{aligned}&\mathbb{P}\left[L_t^{(V)}\le x\right]\\&\approx\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right)\le x\right]\end{aligned} \tag{5.4}$$

and if $\rho_S\neq 0$, then for $x\le JS_0=V_0$, it holds for large J that

$$\begin{aligned}&\mathbb{P}\left[L_t^{(V)}\le x\right]\\&\approx 1-\sum_{k=0}^{m}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)\mathbb{P}\left[N_t^{(m)}=k\right]\end{aligned} \tag{5.5}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable.

Proof. From the construction in Definition 4.1, we know that $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ are $J+1$ independent Brownian motions, and that for each j the jump terms $U_{1,j}, \dots, U_{m,j}$ are also independent of the processes $W_{t,0}, W_{t,1}, \dots, W_{t,J}$ and the default counting process $N_t^{(m)}$. Hence, for a fixed t, conditional on the pair $W_{t,0}, N_t^{(m)}$, then $S_{t,1}, \dots, S_{t,J}$ will be an i.i.d sequence, and therefore a conditional version of the law of large numbers implies that

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^{J} S_{t,j} = \mathbb{E}\left[S_{t,j} \,|\, W_{t,0}, N_t^{(m)} \right] \quad \text{a.s. under} \quad \mathbb{P}\left[\cdot \,|\, W_{t,0}, N_t^{(m)} \right] \tag{5.6}$$

where the subindex j in $\mathbb{E}\left[S_{t,j} \,|\, W_{t,0}, N_t^{(m)} \right]$ on the right-hand side of (5.6) could be any positive integer due to the exchangeability of $S_{t,1}, \dots, S_{t,J}$. Next, by Definition 4.1 together with (4.34), we have that

$$\begin{aligned}
&\mathbb{E}\left[S_{t,j} \,|\, W_{t,0}, N_t^{(m)} \right] \\
&= \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\left(\rho_S W_{t,0} + \sqrt{1-\rho_S^2} W_{t,j} \right) - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\
&= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\rho_S W_{t,0} \right) \mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right].
\end{aligned} \tag{5.7}$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\
&\quad = \exp\left(\frac{\sigma^2\left(1-\rho_S^2\right) t}{2} \right) \mathbb{E}\left[\exp\left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right]
\end{aligned} \tag{5.8}$$

since

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} - \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^{m} \right] \middle| W_{t,0}, N_t^{(m)} \right] \\
&= \mathbb{E}\left[\exp\left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} \right) \middle| W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^{m} \right] \middle| W_{t,0}, N_t^{(m)} \right] \\
&= \mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2} W_{t,j} \right) \right] \mathbb{E}\left[\exp\left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| W_{t,0}, N_t^{(m)} \right] \\
&= \exp\left(\frac{\sigma^2\left(1-\rho_S^2\right) t}{2} \right) \mathbb{E}\left[\exp\left(- \sum_{n=1}^{N_t^{(m)}} U_{n,j} \right) \middle| N_t^{(m)} \right]
\end{aligned} \tag{5.9}$$

where the third equality in (5.9) follows from the fact that $W_{t,j}$ is independent of $W_{t,0}, N_t^{(m)}, \{U_{n,j}\}_{n=1}^m$, and the fourth equality in (5.9) is due to that $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ is independent of $W_{t,0}$, see e.g. 9.7(k) on p.88 in [52], and due to standard computations of $\mathbb{E}\left[\exp\left(\sigma\sqrt{1-\rho_S^2}W_{t,j}\right)\right]$, which proves (5.8). Next, note that

$$\mathbb{E}\left[\exp\left(-\sum_{n=1}^{N_t^{(m)}} U_{n,j}\right) \middle| N_t^{(m)}\right] = \left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \tag{5.10}$$

since

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\sum_{n=1}^{N_t^{(m)}} U_{n,j}\right) \middle| N_t^{(m)}\right] \\
&= \sum_{k=0}^{m} \mathbb{E}\left[\exp\left(-\sum_{n=1}^{k} U_{n,j}\right) \middle| N_t^{(m)} = k\right] 1_{\left\{N_t^{(m)}=k\right\}} \\
&= \sum_{k=0}^{m} \mathbb{E}\left[\exp\left(-\sum_{n=1}^{k} U_{n,j}\right)\right] 1_{\left\{N_t^{(m)}=k\right\}} \\
&= \sum_{k=0}^{m} \left(\frac{\eta}{\eta+1}\right)^{k} 1_{\left\{N_t^{(m)}=k\right\}} \\
&= \left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}
\end{aligned} \tag{5.11}$$

where the second equality in (5.11) is due to that $\{U_{n,j}\}_{n=1}^m$ are independent of $N_t^{(m)}$, and the third equality in (5.11) follows from (A.2.27) in Theorem 2.14. So, combining (5.7), (5.8), and (5.10) together with some computations then renders that

$$\mathbb{E}\left[S_{t,j} \,|\, W_{t,0}, N_t^{(m)}\right] = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \tag{5.12}$$

and (5.12) in (5.6) finally implies

$$\begin{aligned}
&\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^{J} S_{t,j} \\
&= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \quad \text{a.s. under } \mathbb{P}\left[\,\cdot\,|\, W_{t,0}, N_t^{(m)}\right]
\end{aligned} \tag{5.13}$$

which proves (5.2). The random measure $\mathbb{P}\left[\,\cdot\,|\, W_{t,0}, N_t^{(m)}\right]$ is constructed from the probability measure $\mathbb{P}$ used in this paper, and in particular Definition 4.1, so (5.13) then implies that $\frac{1}{J}\sum_{j=1}^{J} S_{t,j}$ converges weakly (i.e in distribution) to $S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)$ under the probability measure $\mathbb{P}$ when $J \to \infty$. To see this, note that

$$\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J} S_{t,j} \le x\right] = \mathbb{E}\left[\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J} S_{t,j} \le x \middle| W_{t,0}, N_t^{(m)}\right]\right] \tag{5.14}$$

and (5.13) imply that

$$\begin{aligned}&\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J} S_{t,j} \leq x \,\middle|\, W_{t,0}, N_t^{(m)}\right]\\&\to \mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \leq x \,\middle|\, W_{t,0}, N_t^{(m)}\right]\end{aligned} \tag{5.15}$$

as $J \to \infty$. Hence, (5.14)-(5.15) together with the law of iterated expectations then renders

$$\begin{aligned}&\mathbb{P}\left[\frac{1}{J}\sum_{j=1}^{J} S_{t,j} \leq x\right]\\&\to \mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \leq x\right] \quad \text{as } J \to \infty\end{aligned} \tag{5.16}$$

which proves (5.3). Thus, if J is large, then (5.16) implies that

$$\sum_{j=1}^{J} S_{t,j} \;\overset{d}{\approx}\Big|_{\mathbb{P}}\; JS_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}} \qquad \text{for large } J \tag{5.17}$$

where $\overset{d}{\approx}\Big|_{\mathbb{P}}$ means "approximately equal in distribution under the probability measure $\mathbb{P}$. Next, from the definition of the portfolio value V_t in (4.6) and the portfolio loss process $L_t^{(V)}$ in (4.9) together with the fact that $S_{t,0} = S_0$ for all stocks due to condition (4.34), we get that

$$L_t^{(V)} = V_0 - V_t = \sum_{j=1}^{J} S_{t,0} - \sum_{j=1}^{J} S_{t,j} = JS_0 - \sum_{j=1}^{J} S_{t,j} \tag{5.18}$$

so (5.17) and (5.18) with some simple calculations then imply that

$$\begin{aligned}&\mathbb{P}\left[L_t^{(V)} \leq x\right]\\&\approx \mathbb{P}\left[JS_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right) \leq x\right] \quad \text{for large } J\end{aligned} \tag{5.19}$$

which proves (5.4).

Next, we want to find an more explicit expression of the right-hand side of (5.19). First, we note that

$$\begin{aligned}&\mathbb{P}\left[JS_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right) \leq x\right]\\&= \sum_{k=0}^{m}\mathbb{P}\left[JS_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^{k}\right) \leq x \,\middle|\, N_t^{(m)} = k\right]\\&\quad \times \mathbb{P}\left[N_t^{(m)} = k\right].\end{aligned} \tag{5.20}$$

Since $W_{t,0}$ is independent of $N_t^{(m)}$, we get

$$\begin{aligned}
&\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^k\right)\leq x\,\middle|\,N_t^{(m)}=k\right]\\
&=\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^k\right)\leq x\right]
\end{aligned}\tag{5.21}$$

and assuming $\rho_S\neq 0$, some calculations then render that

$$\begin{aligned}
&\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta}{\eta+1}\right)^k\right)\leq x\right]\\
&=1-\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)
\end{aligned}\tag{5.22}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. So, combining (5.20)-(5.22) and inserting these expressions into (5.19) finally yields

$$\begin{aligned}
&\mathbb{P}\left[L_t^{(V)}\leq x\right]\\
&\approx 1-\sum_{k=0}^{m}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)\mathbb{P}\left[N_t^{(m)}=k\right]\quad\text{for large } J
\end{aligned}$$

which proves (5.5), and this concludes the theorem. □

Next, we make some remarks on the results in Theorem 5.2.

Remark 5.3. First, we note from (5.2) in Theorem 5.2 that, when conditioning on $W_{t,0}, N_t^{(m)}$ and then studying the limit of $\frac{1}{J}\sum_{j=1}^{J}S_{t,j}$ when $J\to\infty$, we see that the individual diffusions $W_{t,j}$ as well as the individual jump terms $U_{n,j}$ vanish. Only the effect of $W_{t,0}$ and $N_t^{(m)}$ remains in the limit of $\frac{1}{J}\sum_{j=1}^{J}S_{t,j}$ on a simple form as stated in Equation (5.2). Second, if $\rho_S=0$, meaning that there is no correlation through the factor process $W_{t,0}$ in the diffusion part among the stocks, then (5.2) collapses into

$$\lim_{J\to\infty}\frac{1}{J}\sum_{j=1}^{J}S_{t,j}=S_0e^{\mu t}\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\quad\text{a.s. under }\mathbb{P}\left[\cdot\,|\,N_t^{(m)}\right]\tag{5.23}$$

where the right-hand side of (5.23) is a piecewise deterministic process with jumps at the default times $\tau_1,\dots,\tau_m$. If "$\eta=\infty$" so that $U_{n,j}=0$ for all pairs n,j (see also Remark 2.11), and if $\rho_S=0$, then (5.23) reduces to the "standard" law of large numbers under the measure $\mathbb{P}$ since Remark 4.10 with $\rho_S=0$ implies that $S_{t,j}=S_{t,j}^{(\mathrm{BS})}$ for all companies $\mathbf{A}_j$, and $S_{t,1},\dots,S_{t,J}$ will be an i.i.d sequence. This observation together with Equation (2.26) gives $\mathbb{E}\left[S_{t,j}^{(\mathrm{BS})}\right]=S_0e^{\mu t}$ which is the right-hand side of (5.23) without the point process $N_t^{(m)}$, since $U_{n,j}=0$ for all n

and j, that is,

$$\lim_{J\to\infty}\frac{1}{J}\sum_{j=1}^{J}S_{t,j}=S_0e^{\mu t}\quad\mathbb{P}\text{ -a.s.}$$

and this is just the (strong) law of large numbers under the measure $\mathbb{P}$ since $S_{t,1},\dots,S_{t,J}$ is an i.i.d sequence.

For $\rho_S\neq 0$, define $F^{\text{LPA}}_{L_t^{(V)}}(x)$ as

$$F^{\text{LPA}}_{L_t^{(V)}}(x)=1-\sum_{k=0}^{m}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)\mathbb{P}\left[N_t^{(m)}=k\right].\tag{5.24}$$

Then, if $\rho_S\neq 0$, the large portfolio approximation formula (5.5) in Theorem 5.2 implies that

$$\mathbb{P}\left[L_t^{(V)}\leq x\right]\approx F^{\text{LPA}}_{L_t^{(V)}}(x)\quad\text{for large }J\,.\tag{5.25}$$

Note that $F^{\text{LPA}}_{L_t^{(V)}}(x)$ in (5.24) is exactly equal to the right-hand side of (5.4). From the probability in the right hand side of (5.4), it is clear that this probability will be one for $x>V_0=JS_0$, and then $F^{\text{LPA}}_{L_t^{(V)}}(x)=1$ for $x>V_0=JS_0$. To see this, note that for each k we have that

$$\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)\to-\infty\quad\text{ as }\quad x\uparrow JS_0=V_0\,.$$

so for each k we get

$$\lim_{x\uparrow JS_0}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right)-\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right)=0$$

and this observation in (5.24) implies that

$$\lim_{x\uparrow JS_0}F^{\text{LPA}}_{L_t^{(V)}}(x)=1\,.\tag{5.26}$$

Hence, in view of (5.24) and (5.26), the distribution $F^{\text{LPA}}_{L_t^{(V)}}(x)$ is only defined for $x\leq V_0=JS_0$. Consequently, our LPA approximation formula in (5.25) implies that $F^{\text{LPA}}_{L_t^{(V)}}(x)=1$ for $x>V_0=JS_0$, that is, for any time point t, the loss will never be bigger than V_0, which is financially correct given our model setup, while the distribution for the linearized portfolio loss $L_t^{\Delta V}$ discussed in Section 4 can produce losses bigger than $V_0=JS_0$ when t increases.

Here, we note that the distribution function $F^{\text{LPA}}_{L_t^{(V)}}(x)$ defined in (5.24) and used on the the right-hand side of (5.5) in Theorem 5.2 will be much easier to evaluate than the corresponding distribution for the "small time" linear approximation $L_t^{\Delta V}$ to $L_t^{(V)}$, where $\mathbb{P}\left[L_t^{\Delta V}\leq x\right]$ is given by (4.18) in Theorem 4.8. More specifically, the expression for $\mathbb{P}\left[L_t^{\Delta V}\leq x\right]$ in (4.18) will for each $k\geq 1$ in the sum involve computations of an integral given by (4.19) in Theorem 4.8, while the corresponding terms in the sum for $F^{\text{LPA}}_{L_t^{(V)}}(x)$ in (5.24) simply involves an evaluation of the distribution function to a standard normal random variable for each k in the sum.

However, we remind that $\mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ works for heterogeneous stock portfolios with an arbitrary number of stocks J, in particular smaller J, while the approximation of $\mathbb{P}\left[L_t^{(V)} \leq x\right]$ via $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ in (5.25) is only feasible for large stock portfolio sizes J. On the other hand, $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ works for arbitrary time points t, while $\mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ is only a good approximation to $\mathbb{P}\left[L_t^{(V)} \leq x\right]$ for small time points t.

Let $\mathrm{VaR}_\alpha\left(L_t^{(V)}\right)$ defined as in (4.10) be the Value-at-Risk for the stock portfolio loss $L_t^{(V)}$ with confidence level α. By using the large portfolio approximation formula (5.5) in Theorem 5.2, that is, relation (5.25), we can for large J find an approximation to $\mathrm{VaR}_\alpha\left(L_t^{(V)}\right)$ which then is given as the unique solution x^* to the equation $F^{\mathrm{LPA}}_{L_t^{(V)}}(x^*) = \alpha$, that is

$$\mathrm{VaR}_\alpha\left(L_t^{(V)}\right) \approx (F^{-1})^{\mathrm{LPA}}_{L_t^{(V)}}(\alpha) \quad \text{for large } J \tag{5.27}$$

where $(F^{-1})^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ denotes the inverse function to the function $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ defined in (5.24). Since $F^{\mathrm{LPA}}_{L_t^{(V)}}(x) = 1$ for $x > V_0 = JS_0$, we see that (5.27) can never produce a VaR value bigger than V_0, contrary to the linearized portfolio loss VaR-values.

Just as in Theorem 2.14 and Theorem 4.8, we once again remark that the formula in (5.5) in Theorem 5.2 and computations as in (5.27) require efficient and quick methods of computing the number of default distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$.

In the case when there are no jumps in the stock prices at the defaults of the exogenous group of defaultable entities in Definition 4.1, i.e. when "$\eta = \infty$" so that $U_{n,j} = 0$ for all pairs n, j (see also Remark 2.11), and thus $S_{t,j} = S_{t,j}^{(\mathrm{BS})}$ for all companies $\mathbf{A}_j$ where $S_{t,j}^{(\mathrm{BS})}$ is given by (4.39) in Remark 4.10, and if $\rho_S \neq 0$, then (5.2) in Theorem 5.2 will reduce to

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^{J} S_{t,j} = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right) \quad \text{a.s. under } \mathbb{P}\left[\,\cdot \mid W_{t,0}\right]. \tag{5.28}$$

Hence, from (5.28) and using the same arguments as in Theorem 5.2, we then have that

$$\begin{aligned}
&\mathbb{P}\left[L_t^{(V)} \leq x\right] \\
&\approx \mathbb{P}\left[JS_0\left(1 - \exp\left(\left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t + \sigma\rho_S W_{t,0}\right)\right) \leq x\right] \quad \text{for large } J\,.
\end{aligned} \tag{5.29}$$

We also note that the right-hand side of (5.28) is of the exact same form as the stock price $S_t^{(\mathrm{BS})}$ in the Black-Scholes model for a single stock, under the real probability measure $\mathbb{P}$ given in (2.25), but now with the volatility $\sigma\rho_S$ instead of σ as in (2.25). Hence, for large J, the loss process $L_t^{(V)}$ will for the case when $U_{n,j} = 0$ for all n, j behave as the loss process for one single stock which follows the Black-Scholes dynamics with volatility $\sigma\rho_S$, drift μ, and initial value JS_0. From Equation (2.34) in Section 2 together with the large portfolio approximation in (5.29), in the case with no jumps in the stock price, we therefore get that

$$\begin{aligned}&\mathrm{VaR}_\alpha\left(L_t^{(V)}\right)\\&\approx JS_0\left(1-\exp\left(\sigma\rho_S\sqrt{t}\Phi^{-1}\left(1-\alpha\right)+\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t\right)\right)\quad\text{for large } J\,.\end{aligned}\tag{5.30}$$

In our numerical studies in Section 7 and 8, we will use the "Black-Scholes" LPA VaR formula in (5.30) as a benchmark for the VaR formulas obtained when using the LPA loss distribution (5.5) in Theorem 5.2 when the stock prices have jumps and are exchangeable.

6. **Comparison against Kou model with only negative jumps.** In Sections 4 and 5, we considered the case where the jumps in the stock prices where triggered by default from an exogenous group of m entities $\mathbf{C}_1,\ldots,\mathbf{C}_m$. In our numerical studies, we will use the stock price models from Sections 4 and 5 to compute VaR for stock portfolios and compare these VaR-numbers with corresponding values coming from an equity model without jumps, that is, a Black-Scholes portfolio setting under the real probability measure. However, comparing our jump-at-default model only with a non-jump model will in our view not be fully fair. We believe it is equally important to compare our stock price model containing jumps at defaults with other equity models that include jumps in the stock price driven by, e.g., a Poisson process. We will therefore in this section briefly outline some important quantities derived from the Kou model, [32], restricted to only having negative jumps. These quantities will then be used in Section 10 when comparing VaR-values coming from our jump-at-defaults model in Section 5 with VaR-quantities from the Kou model having only negative jumps which are driven by a Poisson process outlined in this section.

Let $S_{t,j}^{(\kappa)}$ be the stock price in Definition 4.1, but where the default counting process $N_t^{(m)}$ is replaced by a Poisson process N_t with intensity $\lambda_\kappa>0$, so that $S_{t,j}^{(\kappa)}$ is given by

$$S_{t,j}^{(\kappa)}=S_{0,j}\exp\left(\left(\mu_j-\frac{1}{2}\sigma_j^2\right)t+\sigma_j\left(\rho_{S,j}W_{t,0}+\sqrt{1-\rho_{S,j}^2}W_{t,j}\right)-\sum_{n=1}^{N_t}U_{n,j}\right)\tag{6.1}$$

and where the rest of the terms in (6.1) are the same as in Definition 4.1. The stock prices $S_{t,j}^{(\kappa)}$ in (6.1) will be same as in [32], but here with only negative jumps, and where we extend [32] to a portfolio setting so that the $S_{t,j}^{(\kappa)}$ are correlated via the factor process $W_{t,0}$ and the Poisson process N_t. Next, we define $V_t^{(\kappa)}$ and $L_t^{V,\kappa}$ as

$$V_t^{(\kappa)}=\sum_{j=1}^{J}w_jS_{t,j}^{(\kappa)}\qquad\text{and}\qquad L_t^{V,\kappa}=-\left(V_t^{(\kappa)}-V_0^{(\kappa)}\right)\tag{6.2}$$

Given the two stock portfolio frameworks $S_{t,j}^{(\kappa)}$ modeled as in (6.1) and $S_{t,j}$ given by Definition 4.1, it is interesting to compare these two models. To this end, we first state the following corollary to Theorem 2.14.

Corollary 6.1. *Let $S_{t,j}^{(\kappa)}$ be a stock price under the real probability measure $\mathbb{P}$ defined as in (6.1). Then, with notation as above, we have that*

$$\mathbb{E}\left[S_{t,j}^{(\kappa)}\,\middle|\,N_t\right] = S_{0,j}e^{\mu_j t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^{N_t} \quad \textit{where}$$

$$\mathbb{E}\left[S_t^{(\kappa)}\,\middle|\,N_t = k\right] = S_{0,j}e^{\mu_j t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^{k} \tag{6.3}$$

for $k = 0, 1, 2, \ldots, m$ and

$$\mathbb{E}\left[S_{t,j}^{(\kappa)}\right] = S_{0,j}\exp\left(\left(\mu_j - \frac{\lambda_\kappa}{\eta_\kappa+1}\right)t\right). \tag{6.4}$$

Furthermore, if the stock portfolio is homogeneous so that condition (4.34) is satisfied with the weights $w_j = 1$ in (6.2), then for large J we have that

$$\mathbb{P}\left[L_t^{V,\kappa} \le x\right]$$
$$\approx 1 - \sum_{k=0}^{\infty} e^{-\lambda_\kappa t}\frac{(\lambda_\kappa t)^k}{k!}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta_\kappa+1}{\eta_\kappa}\right)^k\right) - \left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right) \tag{6.5}$$

where the rest of the notation in (6.5) is same as in Theorem 5.2.

A proof of Corollary 6.1 is given in Subsection A.4 of Appendix A.

In Sections 7, 8, and 9, we will later compare the stock portfolio VaR in the model with jumps at defaults given by Definition 4.1 with the corresponding Black-Scholes portfolio model without jumps. But, we will also benchmark our model with the model in [32] with only negative jumps, that is, the model $S_{t,j}^{(\kappa)}$ given by (6.1), see Section 10. For the comparison of $S_{t,j}$ in Definition 4.1 with the Kou model, [32], we will consider a homogeneous stock portfolio so that condition (4.34) is satisfied, that is, $S_{0,j} = S_0$, $\mu_j = \mu$, $\sigma_j = \mu$, and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$ so that the stock prices $S_{t,1}^{(\kappa)}, S_{t,2}^{(\kappa)}, \ldots, S_{t,J}^{(\kappa)}$ become exchangeable. Now, assume that we want to calibrate the parameters λ_κ and η_κ in the model for $S_{t,j}^{(\kappa)}$ given by (6.1). There are several different ways of calibrating λ_κ and η_κ. If we want to compare the two models $S_{t,j}^{(\kappa)}$ and $S_{t,j}$, then we can, for example, assume that for two arbitrary fixed time points T and $\tilde{T}$, it will hold that

$$\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0 = \mathbb{E}\left[S_{T,j}\right] \quad \text{and} \quad \mathbb{E}\left[N_{\tilde{T}}\right] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right] \tag{6.6}$$

where we remark that it is possible to let $\tilde{T} = T$. Note that the condition $\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0$ implies, just as in (2.32), that the downward jumps in the Kou model $S_t^{(\kappa)}$ at the jump times of the Poisson process N_t "wipe" out the expected log-growth for a corresponding Black-Scholes model with drift μ up to time T; see also Equation (6.4). Furthermore, we observe that $\mathbb{E}\left[N_{\tilde{T}}\right] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right]$ means that the expected number of jumps by the point processes N_t and $N_t^{(m)}$ will be the same up to time $\tilde{T}$, which for $N_t^{(m)}$ is the same as saying that the expected number of defaults in the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$ will be given by $\mathbb{E}\left[N_{\tilde{T}}\right] = \lambda_\kappa\tilde{T}$. If the default times $\tau_1, \tau_2 \ldots, \tau_m$

for $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable with default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$, then the condition $\mathbb{E}[N_{\tilde{T}}] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right]$ can be reformulated as

$$\lambda_\kappa \tilde{T} = mF(\tilde{T}). \tag{6.7}$$

For $S_{t,j}$ in Definition 4.1 under condition (4.34), and from (2.32), we see that the condition $\mathbb{E}\left[S_T^{(\kappa)}\right] = S_0$ gives a non-linear equation for finding the jump parameter η_κ, and in a similar way we can use (6.4) in Corollary 6.1 to conclude that

$$\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0 \quad \text{if and only if} \quad \exp\left(\left(\mu - \frac{\lambda_\kappa}{\eta_\kappa + 1}\right) T\right) = 1$$

$$\text{and thus} \quad \frac{\lambda_\kappa}{\eta_\kappa + 1} = \mu. \tag{6.8}$$

Hence, in view of (6.7) and (6.8), condition (6.6) in an exchangeable credit portfolio model for the default times $\tau_1, \tau_2 \ldots, \tau_m$ to $\mathbf{C}_1, \ldots, \mathbf{C}_m$ can then be reformulated as

$$\lambda_\kappa \tilde{T} = mF(\tilde{T}) \quad \text{and} \quad \frac{\lambda_\kappa}{\eta_\kappa + 1} = \mu. \tag{6.9}$$

If μ is known, and if the parameters of the default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$ also are known, then condition (6.6), or equivalently (6.9), will give us two unknown parameters, λ_κ and η_κ, and two equations, which often will lead to semi-explicit or explicit solutions for λ_κ and η_κ. For example, if the exchangeable default times $\tau_1, \tau_2 \ldots, \tau_m$ have constant default intensity λ so that $F(t) = \mathbb{P}[\tau_i \leq t] = 1 - e^{-\lambda t}$, then (6.9) implies that λ_κ and η_κ are given by

$$\lambda_\kappa = \frac{m\left(1 - e^{-\lambda \tilde{T}}\right)}{\tilde{T}} \quad \text{and} \quad \eta_\kappa = \frac{\lambda_\kappa}{\mu} - 1. \tag{6.10}$$

So, λ_κ and η_κ in (6.10) are thus equivalent with the conditions in (6.6) which are financially and intuitively clear. In Section 10, we will use λ_κ and η_κ in (6.10) when computing and comparing VaR-values coming from our jump-at-defaults model in Section 5 with VaR-quantities from the Kou model in (6.1) having only negative jumps which are exponentially distributed with parameter η_κ where the jumps are driven by a Poisson process with intensity λ_κ. Note that condition (6.10) is independent of what type of credit portfolio model we use for the default times as long as $F(t) = \mathbb{P}[\tau_i \leq t] = 1 - e^{-\lambda t}$ for all default times.

7. **Numerical examples when the default times have CIR intensities.** In this section we will study Value-at-Risk for the loss $L_t^{(S)} = -(S_t - S_0)$ for one single stock when the stock price S_t is given by Definition 2.1 under the real probability measure $\mathbb{P}$. Throughout this section we assume that the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditional independent, and have default intensities $\lambda_{t,i} = \lambda_t$ the same for all entities where λ_t is a CIR-process. Furthermore, the jumps $\tilde{V}_1, \ldots, \tilde{V}_m$ in S_t at the defaults $\tau_1, \tau_2 \ldots, \tau_m$ are distributed as $V_1, \ldots, V_m$ in Definition 2.10. In Subsection 7.1, we first motivate the rationale of the numerical values for the parameters, and also study other related quantizes such as the number of defaults distribution for the group of defaultable entities. Then, in Subsection 7.2 we study Value-at-Risk for the loss of one individual stock with price under the real probability measure $\mathbb{P}$ in a credit portfolio model with parameters as discussed in Subsection 7.1. Finally, in Subsection 7.3 we give some very important

and useful remarks on the numerical computation of the loss distribution. The observations done in Subsection 7.3 will also hold for the loss distributions derived in Section 4 and Section 5, and for the credit portfolio model studies in Section 8.

7.1. **The parameters and related quantities.** In this subsection, we discuss the modeling setup and its parameters that will hold in the rest of the section, and present some related quantities such as, e.g., the distribution of the number of defaults $\left(\mathbb{P}\left[N_t^{(m)}=k\right]\right)_{k=0}^{m}$ for our model.

As mentioned in Section 1, in this paper we will not focus on how to estimate the involved parameters describing the stock model, including the parameters for the defaultable entities affecting the equity prices. Instead, the main goal of this paper is to derive analytical stock portfolio quantities in our equity-credit hybrid model, and then use these to numerically study the time evolution of VaR for equity portfolios and compare the VaR-numbers with corresponding values coming from alternative models, such as the Kou model and the Black-Scholes model. However, in this subsection we will motivate the rationale behind the choice of the stock price parameters μ, σ, and the one-year default probability for the defaultable entities connected to the default intensity parameters. Since the numerical values of μ, σ, and the one-year default probability will also be used in Sections 8 - 10, we will then simply refer to this subsection for motivation of the choice of the parameters μ, σ, and the one-year default probability.

In the rest of this section, we assume that the default times $\tau_1,\tau_2\dots,\tau_m$ to the entities $\mathbf{C}_1,\dots,\mathbf{C}_m$ are exchangeable, conditionally independent, and have default intensities $\lambda_{t,i}=\lambda_t$ the same for all entities. We set $\lambda_t=\lambda_{t,i}$ to be a Cox-Ingersoll-Ross process (CIR-process), that is,

$$d\lambda_t = a_c\left(\mu_c-\lambda_t\right)dt+\sigma_c\sqrt{\lambda_t}dW_t^{(c)} \tag{7.1.1}$$

where $W_t^{(c)}$ is a Brownian motion under the physical probability measure $\mathbb{P}$, independent of the other random variables in S_t. Then, the default times $\tau_1,\dots,\tau_m$ are constructed as

$$\tau_i=\inf\left\{t>0:\int_0^t\lambda_s ds\geq E_i\right\} \tag{7.1.2}$$

where $E_1,\dots,E_m$ is an i.i.d sequence of exponentially distributed random variables all with parameter one which are independent of $W_t^{(c)}$. From the construction (7.1.2), one can show that $\tau_1,\dots,\tau_m$ are conditionally independent given the trajectory of $(W_t^{(c)})_{t\geq 0}$. Furthermore, the marginal default distribution $F(t)=\mathbb{P}[\tau_i\leq t]$ is expressed as

$$F(t)=\mathbb{P}[\tau_i\leq t]=1-\mathbb{E}\left[e^{-\int_0^t\lambda_s\,ds}\right] \tag{7.1.3}$$

and is the same for all entities $\mathbf{C}_1,\dots,\mathbf{C}_m$ due to the exchangeability, where the quantity $\mathbb{E}\left[e^{-\int_0^t\lambda_s\,ds}\right]$ has closed formulas; see e.g. [5] or [25]. The construction in (7.1.2)-(7.1.3) can be applied to arbitrary intensities λ_t, and thus not only to a CIR-process. From a practical point of view, we want to have analytical expressions of the default distribution $F(t)$ in (7.1.3). Another example of intensity which gives analytical formulas for $F(t)$ is a shot-noise model as presented in, e.g., [26]. The construction in (7.1.2)-(7.1.3) will also work for heterogeneous credit portfolios, that is, when the intensities $\lambda_{i,t}$ are different among the entities $\mathbf{C}_1,\dots,\mathbf{C}_m$.

Going back to our stock price model for S_t, we let the jumps $\tilde{V}_1,\dots,\tilde{V}_m$ in S_t at the defaults $\tau_1,\tau_2\dots,\tau_m$ be distributed as $V_1,\dots,V_m$ in Definition 2.10, so $\tilde{V}_i=$

$e^{-\tilde{U}_i} - 1$ where $\tilde{U}_1, \ldots, \tilde{U}_m$ are i.i.d and exponentially distributed with parameter $\eta > 0$. Hence, given the above assumptions, the dynamics of the stock price S_t is the same as in Corollary 2.13 and Theorem 2.14 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and $\tau_1, \tau_2 \ldots, \tau_m$ are exchangeable, conditionally independent, and have intensities $\lambda_{t,i} = \lambda_t$ as in (7.1.1).

The average historic one-year default rate for speculative-grade entities during the period 2000-2023 was 3.64%; see e.g. Table 1 in [50]. Furthermore, from Table 1 in [50], we also observe that the average one-year investment-grade default rate in the same period (2000-2023) is much smaller, and is given as 0.08%. We therefore want to have a one-year default probability in the upper part of the interval $[0.08, 3.64]$, and in our numerical examples we choose a CIR-process in (7.1.1) with the parameters $a_c = 0.6, \mu_c = 0.056, \sigma_c = 0.18$, and $\lambda_0 = 0.0262$, which renders an individual one-year default probability of $0.0329 = 3.29\%$, computed via well-known explicit expressions for the default probability $\mathbb{P}[\tau_i \leq t]$ when τ_i has a CIR-default intensity. We also make sure that $2a_c\mu_c > \sigma_c^2$ so that zero is avoided; see e.g. p.391 in [13]. The above observations motivate the choice of the parameters for the CIR-process and also the one-year default probability, and are all stated in Table 1.

Table 1. The parameters and related quantities for the CIR-process λ_t and the stock price S_t where we let $m = 125$.

λ_t	$\lambda_0 = 0.0262$	$a_c = 0.6$	$\mu_c = 0.056$	$\sigma_c = 0.18$	$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$
S_t	$S_0 = 50$	$\mu = 0.15 = 15\%$	$\sigma = 0.2 = 20\%$	$\eta = 26.71$	$\mathbb{E}[U] = \frac{1}{\eta} = 0.0374 = 3.74\%$

Furthermore, we let the number of defaultable entities be $m = 125$, see Table 1, which is the same number of entities that are found in main CDS indices such as the iTraxx Europe and CDX.NA.IG US. Of course, one can choose a higher value for m, but here we set $m = 125$. In Table 2, we show the expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ for $t = 1, 3, 6, 12, 18, 24$ months when individual default times have CIR-intensities as in Table 1 and where $m = 125$. So, from Table 2 we see that our CIR-intensities implies that we expect, for example, around 2 defaults in six months, 4 defaults in one year, and 6 to 7 defaults in one and a half years. Consequently, this is also the number of jumps that we expect to occur in our stock price up to each of these time points where each jump has the expected size of $\mathbb{E}[U] = \frac{1}{\eta}$. By our assumption of exchangeability, we have that $\mathbb{E}\left[N_t^{(m)}\right] = m\mathbb{P}[\tau_i \leq t]$, so the individual default probabilities at $t = 1, 3, 6, 12, 18, 24$ months are obtained from Table 2 by dividing the numbers for $\mathbb{E}\left[N_t^{(m)}\right]$ with m. In Table 2, we see that after 6 months there is a 0.1% probability of having 25 defaults or more among $\mathbf{C}_1, \ldots, \mathbf{C}_m$, and after 24 months there is a 0.1% probability of 32 defaults or more among $\mathbf{C}_1, \ldots, \mathbf{C}_m$.

Next, we turn to the parameters for the stock price model. We set $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$, see Table 1. First, the motivation for choosing $\mu = 15\%$ follows from the fact that during the 10-year period of 2012 to the end of 2021, the average one-year US stock market return was 14.88% not adjusted for inflation; see e.g. [49]. So, solving for μ in the Black-Scholes model with $\mathbb{E}\left[S_t^{(\mathrm{BS})}\right] = S_0 e^{\mu t} =$

Table 2. The expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ and $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ for $t = 1, 3, 6, 12, 18, 24$ months when individual default times have CIR-intensities as in Table 1 and where $m = 125$.

t (in months)	1	3	6	12	18	24
$\mathbb{E}\left[N_t^{(m)}\right]$	0.2802	0.8818	1.875	4.116	6.596	9.222
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$	20	25	25	25	27	32

$S_0 \cdot 1.1488$ with $t = 1$, or simply by using a first-order expansion of $e^{\mu} \approx 1 + \mu$, after rounding with two significant digits, we get that $\mu = 0.15 = 15\%$. Note that we used the Black-Scholes model to find μ, and one can of course also use our stock price model in Corollary 2.13, but this will lead to a very complicated estimation problem, which is not the topic of this paper. Next, the motivation for the choice of $\sigma = 20\%$ follows from the fact that, during the 9-year period 2012 to the end of 2020, the average one-year US stock market volatility was 16.40% not adjusted for inflation; see e.g. [51]. However, accounting for that the VIX volatility has been higher during the period of 2012 to the end of 2020, we therefore set $\sigma = 20\%$; see e.g. [39].

Finally, the jump parameters η are challenging to estimate. Instead, we chose to express the expected one-year stock value in our model as some fixed known value; specifically, we let $\mathbb{E}\left[S_T\right] = S_0$, which together with the default distribution for $N_t^{(m)}$ will imply a value of η. Hence, η is calibrated so that the defaults from the CIR-model "wipe" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. Thus, the jump parameter η is calibrated so that, for $T = 1$ year, we have

$$\mathbb{E}\left[S_T\right] = S_0 \quad \text{or equivalently} \quad \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_T^{(m)}}\right] = e^{-\mu T} \quad \text{for } T = 1 \tag{7.1.4}$$

see also Equation (2.32) in Section 2. With the above parameters, we get that $\eta = 26.71$ via a numerical solver so that $\mathbb{E}\left[U_i\right] = \frac{1}{\eta} = 0.0374$, see Table 1. Alternatively, we could just pick an ad-hoc value of η, but we find it much more economically intuitive to use the condition (7.1.4) to find a numerical value of η. More generally, we can use condition (2.30) or equivalently condition (2.31) with an arbitrary value of $\beta \in (0, 1)$ to find the implied η-parameter.

The above numerical choices of μ, σ, and the one-year default probability will consistently be used in the rest of this paper, and therefore in Sections 8 - 10 we will just state these values for μ, σ, and the one-year default probability in tables and refer to this subsection for more motivations of the choice. Furthermore, the method of finding η via condition (7.1.4) is chosen to be the same in Sections 8 - 10, however the numerical value of η will be different, since in Sections 8 - 10 we consider other credit portfolio models for the defaultable entities, thus leading to other numerical values of the distribution of $N_t^{(m)}$ used in (7.1.4).

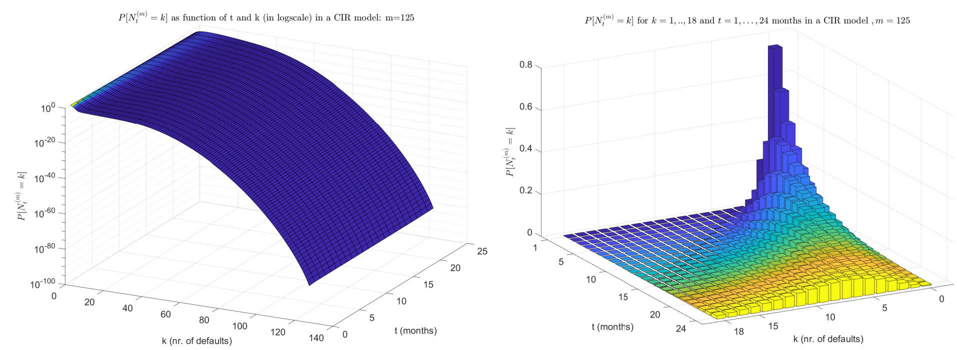

FIGURE 1. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$ for $t=1,2,..24$ months when individual default times have CIR-intensities as in Table 1 where $m=125$. **Left panel:** in log-scale for $k=0,...,125$. **Right panel:** for $k=0,...,18$. The plots in the two panels are viewed from different angles.

From Theorem 2.14 and the definition of VaR, we know that in order to compute $\mathrm{VaR}_\alpha\left(L_t^{(S)}\right)$, we need to compute the distribution of the number of defaults $\left(\mathbb{P}\left[N_t^{(m)}=k\right]\right)_{k=0}^{m}$. Finding efficient numerical methods for $\mathbb{P}\left[N_t^{(m)}=k\right]$ is a non-trivial problem. We will in this paper use the method developed in [25] to find $\mathbb{P}\left[N_t^{(m)}=k\right]$, which is based on saddlepoint methods for exchangeable, conditionally independent credit portfolio models, and works both for intensity based frameworks as well as in factor copula settings. To find $\mathbb{P}\left[N_t^{(m)}=k\right]$ in the intensity based case, we need the density $f_{Z_t}(z)$ of the random variable $Z_t=\int_0^t\lambda_u du$ where λ_t is a CIR-process defined as in (7.1.1). Details of how to find $f_{Z_t}(z)$ as well as numerical graphs of $f_{Z_t}(z)$ are found in, e.g., [25].

With the CIR-parameters parameters in Table 1, we compute $\mathbb{P}\left[N_t^{(m)}=k\right]$ with the saddlepoint method mentioned above, and show in the left panel of Figure 1 plots, for $m=125$, of the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$ in log-scale where $k=0,...,125$ and $t=1,2,\dots,24$ months. Furthermore, the right panel in Figure 1 displays the time evolution of the number of distribution $\mathbb{P}\left[N_t^{(m)}=k\right]$ in normal scale where $k=0,1,\dots,18$ when $m=125$ and $t=1,\dots,24$ months when individual default times have CIR-intensities with parameters the same as in the left panel of Figure 1. The plots in Figure 1 were generated with the saddlepoint algorithms found in [25], and in these figures we write t in months, but the actual computations of $\mathbb{P}\left[N_t^{(m)}=k\right]$ are done with t measured in units of years. So, for example, 2, 6, and 24 months mean that t is given by $t=\frac{2}{12},\frac{6}{12}$ and $t=\frac{24}{12}=2$ in our formulas for the computation of $\mathbb{P}\left[N_t^{(m)}=k\right]$. The same also holds for the results in Table 2.

7.2. **VaR over a 2-year period for one stock when the jumps in the stock price are due to default times with CIR-intensities.** In this subsection we will study Value-at-Risk for the loss $L_t^{(S)} = -(S_t - S_0)$ of one individual stock with price S_t given by Definition 2.1 under the real probability measure $\mathbb{P}$ in a credit portfolio model as discussed in Subsection 7.1. Hence, the stock price S_t has jumps at the default times $\tau_1, \tau_2 \ldots, \tau_m$, which are exchangeable and where the individual default times have CIR-intensities with parameters the same as in Table 1. Furthermore, the jump parameter η is calibrated so that condition (7.1.4) holds and the rest of the parameters for S_t are displayed in Table 1.

In Figures 2-3, we study the time evolution of Value-at-Risk (in % of S_0) of a single stock for $t = 1, 2, \ldots, 24$ months, computed with same stock parameters as in Table 1. More specifically, for $m = 125$, the left panel in Figure 2 displays the time evolution of Value-at-Risk in % of S_0 for $t = 1, 2, \ldots, 24$ months in the case when S_t has jumps coming from default times which have CIR-intensities with parameters the same as in Table 1. The right panel in Figure 2 displays the Black-Scholes case for the stock price, i.e. with no jumps in S_t, which has same drift and volatility parameters as in the left panel.

The interpretation of the results in Figure 2 is as follows: For example, in the left panel of Figure 2, looking at the black line (99%-VaR), we see that for $t = \frac{14}{12}$, that is, after 14 months, then there is a 1% probability of having a loss in the stock which is 50% or bigger of the initial stock price S_0 at time $t = 0$. Similarly, for the red line (99.9%-VaR) in the left panel of Figure 2, at $t = \frac{20}{12}$, that is, 20 months after the starting point $t = 0$, there is 0.1% probability of having a stock loss which is 70% (or bigger) of the starting value S_0 at time $t = 0$. The interpretation of the graphs in the right panel of Figure 2, i.e. the Black-Scholes case, should be done in the same way as in the left panel of Figure 2. Furthermore, in Figure 3 we plot the time evolution of the relative difference of Value-at-Risk (in %) between the case with jumps in the stock price S_t coming from default times which have CIR-intensities with parameters the same as in Table 1, and the standard Black-Scholes case, i.e. without jumps. The rest of the parameters for S_t are the same as in Table 1.

As can be seen in Figure 3, introducing downward jumps in S_t at the default times $\tau_1, \tau_2 \ldots, \tau_m$ which are exchangeable and where the individual default times have CIR-intensities as in Subsection 7.1 will in general increase the Value-at-Risk up to around 50% and much more at a few time points (up to 250%) compared to the Black-Scholes model, and this holds for all three confidence levels $\alpha = 95\%, 99\%$, and 99.9%. For $\alpha = 95\%$, the relative difference (jump-stock model vs. Black-Scholes) is almost linearly increasing in time t. Of course, that the relative VaR difference between the jump vs. non-jump case will increase as shown in Figure 3 is not surprising, but knowing exactly how big the difference actually is as a function of different parameters as well as time t requires the use of somewhat analytical formulas and efficient numerical methods.

7.3. **Some remarks on the numerical computation of the loss distributions.** In this subsection, we give some important remarks on the computation of the loss distribution $F_{L_t^{(S)}}(x)$. The observations done in this subsection will also hold for the loss distributions derived in Section 4 and Section 5, and for the credit portfolio model studies in Section 8.

The computations in the left panel of Figure 2 are done by numerically solving Equation (2.33). From Theorem 2.14, we know that $F_{L_t^{(S)}}(x) = \mathbb{P}\left[L_t^{(S)} \leq x\right]$ is

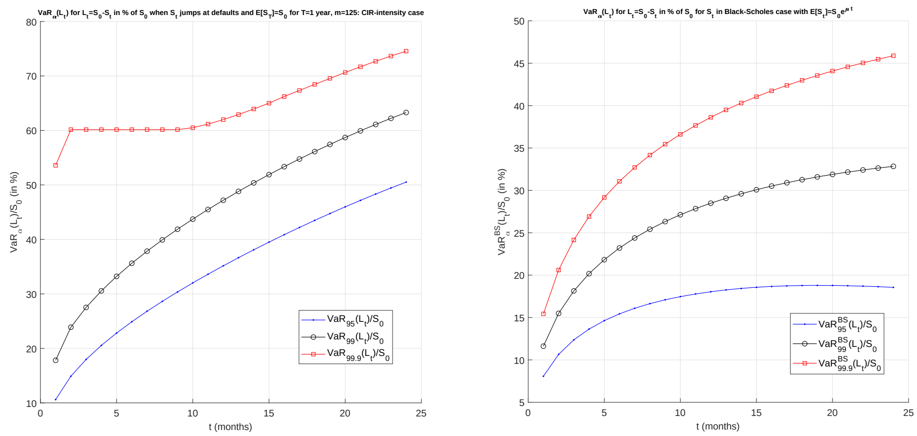

FIGURE 2. $m = 125$: The time evolution of Value-at-Risk (in % of S_0) of a single stock for $t = 1, 2, \ldots, 24$ months. **Left panel:** In the case with jumps in S_t coming from default times which have CIR-intensities with parameters the same as in Table 1. **Right panel:** In the Black-Scholes case, i.e. without jumps, where drift and volatility are the same as in Table 1.

given by

$$F_{L_t^{(S)}}(x) = 1 - \sum_{k=0}^{m} \Psi_k \left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta \right) \mathbb{P}\left[N_t^{(m)} = k \right] \tag{7.3.1}$$

where the mappings $\Psi_k(x, t, \mu, \sigma, u, \eta)$ satisfy $0 \leq \Psi_k(x, t, \mu, \sigma, u, \eta) \leq 1$ and are defined in (2.19)-(2.20). By looking at, e.g., the left panel in Figure 1, but also in the left panels of Figures 4, 11, and 13, we see that the probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ are extremely small for moderate and large integers k for most time points t. For example, in the left panels of Figure 1, we have that $\mathbb{P}\left[N_t^{(m)} = k\right] < 10^{-14}$ for $k \geq 65$ at all time points t, and $\mathbb{P}\left[N_t^{(m)} = k\right] < 10^{-28}$ for $k \geq 85$ at all t. These observations mean that we do not have to compute all the terms in the sum for $F_{L_t^{(S)}}(x)$ given by (7.3.1), but still have a very accurate approximation to $F_{L_t^{(S)}}(x)$ in the truncated sum. For example, let ε be a very small positive constant, e.g. $\varepsilon \leq 10^{-9}$. Then, for each fixed t, there exists a subsequence $k_0, k_1, k_2, \ldots, k_{m_t(\varepsilon)}$ of the integers $0, 1, 2, \ldots, m$ such that

$$\sum_{j=0}^{m_t(\varepsilon)} \mathbb{P}\left[N_t^{(m)} = k_j \right] \geq 1 - \varepsilon\,. \tag{7.3.2}$$

In the credit portfolio models used in this paper, the subsequence $k_0, k_1, k_2, \ldots,$ $k_{m_t(\varepsilon)}$ can always be chosen in the form $0, 1, \ldots, m_t(\varepsilon)$, that is, $k_j = j$ for $j =$

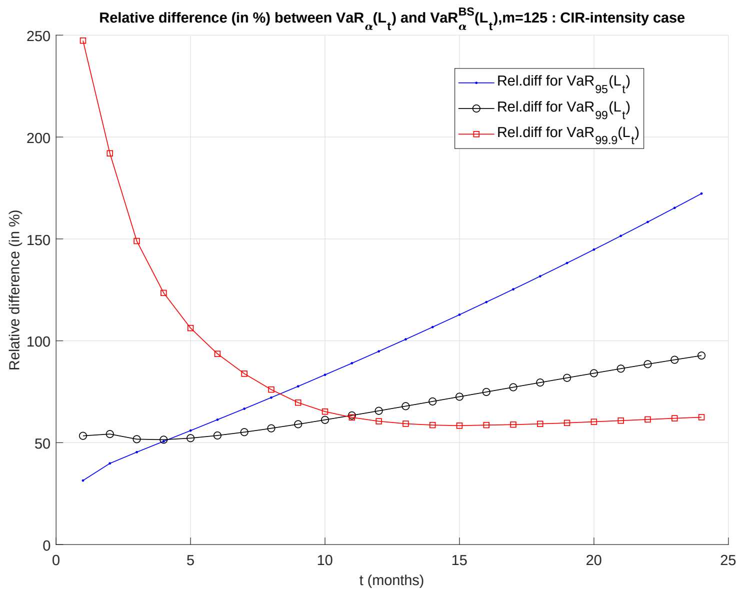

FIGURE 3. The time evolution of the relative difference of Value-at-Risk (in %) between the case of with jumps in the stock price S_t coming from default times which have CIR-intensities with parameters the same as in Table 1, and the standard Black-Scholes case, i.e. without jumps. The relative difference is measured with respect to the Black-Scholes case. The rest of the parameters for S_t are the same as in Table 1.

$0, 1, \ldots, m_t(\varepsilon)$, so that (7.3.2) can be rewritten as

$$\sum_{k=0}^{m_t(\varepsilon)} \mathbb{P}\left[N_t^{(m)} = k\right] \geq 1 - \varepsilon \quad \text{and thus} \quad \sum_{k=m_t(\varepsilon)+1}^{m} \mathbb{P}\left[N_t^{(m)} = k\right] < \varepsilon \tag{7.3.3}$$

where it obviously holds that $m_t(\varepsilon) \leq m$ for any $0 < \varepsilon < 1$ and at all time points t. Typically, for the credit portfolio models studied in this paper, it will often (but not always) hold that $m_t(\varepsilon) << m$ for most time points t. Given an arbitrary number $0 < \varepsilon < 1$, and for a fixed t, we can in view of the above observations define the function $F^{\varepsilon}_{L_t^{(S)}}(x)$ as

$$F^{\varepsilon}_{L_t^{(S)}}(x) = 1 - \sum_{k=0}^{m_t(\varepsilon)} \Psi_k\left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta\right) \mathbb{P}\left[N_t^{(m)} = k\right] \tag{7.3.4}$$

where the rest of the parameters and mappings are defined as in (7.3.1). Then, (7.3.1), (7.3.3), and (7.3.4) together with the triangle inequality imply that

$$\left| F_{L_t^{(S)}}(x) - F^{\varepsilon}_{L_t^{(S)}}(x) \right| \leq \varepsilon \quad \text{for all } x \in \mathbb{R} \tag{7.3.5}$$

where in (7.3.5) we also used that $0 \leq \Psi_k(x, t, \mu, \sigma, u, \eta) \leq 1$ for all k. Hence, for small ε, then (7.3.5) implies that $F^{\varepsilon}_{L_t^{(S)}}(x)$ will be a very sharp approximation to the loss distribution $F_{L_t^{(S)}}(x)$ in (7.3.1). Since it will often hold that $m_t(\varepsilon) << m$, computing $F^{\varepsilon}_{L_t^{(S)}}(x)$ will be much faster than computing the exact distribution $F_{L_t^{(S)}}(x)$, while simultaneously having an accuracy of $F^{\varepsilon}_{L_t^{(S)}}(x)$ compared to $F_{L_t^{(S)}}(x)$ that is smaller than ε given relation (7.3.5). Table 3 displays $m_t(\varepsilon)$ for $t = 1, 3, 6, 12, 18, 24$ months, where $m = 125$ and $\varepsilon = 10^{-9}$ when the individual default times have CIR-intensities as in Table 1. Hence, from Table 3 we see that in order to have an

TABLE 3. The upper truncation level $m_t(\varepsilon)$ defined as in (7.3.3) for $t = 1, 3, 6, 12, 18, 24$ months where $m = 125$ and $\varepsilon = 10^{-9}$, when the individual default times have CIR-intensities as in Table 1.

t (in months)	1	3	6	12	18	24
$m_t(\varepsilon)$	53	54	54	54	54	55

accuracy of order $\varepsilon = 10^{-9}$ in our approximation $F^{\varepsilon}_{L_t^{(S)}}(x)$ to the exact distribution $F_{L_t^{(S)}}(x)$ at the time points $t = 1, 3, 6, 12, 18, 24$, we never need to have more than 56 terms in the sum of $F^{\varepsilon}_{L_t^{(S)}}(x)$ compared with 126 terms in $F_{L_t^{(S)}}(x)$ (recall that we start counting from 0, so, e.g., $m_t(\varepsilon) = 55$ means 56 terms in the sum for $F^{\varepsilon}_{L_t^{(S)}}(x)$ etc.). Also, note that for, e.g., 99.9% VaR computations, we will in our numerical solution of Equation (2.33) work with x^*-values so that $F_{L_t^{(S)}}(x^*) = 0.999$. Since we choose $\varepsilon = 10^{-9}$, and since both $F_{L_t^{(S)}}(x)$ and $F^{\varepsilon}_{L_t^{(S)}}(x)$ are continuous mappings in x, and the error-bound in (7.3.5) holds uniformly for all $x \in \mathbb{R}$, then the solution x^*_ε of the equation $F^{\varepsilon}_{L_t^{(S)}}(x^*_\varepsilon) = 0.999$ should therefore be extremely close to the exact VaR solution x^* satisfying $F_{L_t^{(S)}}(x^*) = 0.999$. More specifically, from (7.3.5) we have

$$10^{-9} \geq \left| F_{L_t^{(S)}}(x^*_\varepsilon) - F^{\varepsilon}_{L_t^{(S)}}(x^*_\varepsilon) \right| = \left| F_{L_t^{(S)}}(x^*_\varepsilon) - 0.999 \right|$$

so that the solution x^*_ε of the equation $F^{\varepsilon}_{L_t^{(S)}}(x^*_\varepsilon) = 0.999$ will give a value of $F_{L_t^{(S)}}(x^*_\varepsilon)$ that deviates at most 10^{-9} from $\alpha = 0.999 = 99.9\%$, which is very accurate. Hence, we can therefore approximate the exact 99.9% VaR value x^*, with x^*_ε obtained from solving $F^{\varepsilon}_{L_t^{(S)}}(x^*_\varepsilon) = 0.999$ where the function $F^{\varepsilon}_{L_t^{(S)}}(x)$ is defined as in (7.3.4). Similar arguments obviously hold for the 99% VaR and 95% VaR computations.

Furthermore, note that 56 terms (i.e. $m_t(\varepsilon)+1$) versus 126 terms when $m = 125$ (i.e $m + 1 = 126$) will mean a running time of VaR computations with $F^{\varepsilon}_{L_t^{(S)}}(x)$ more than twice as fast compared with VaR computations for the exact distribution $F_{L_t^{(S)}}(x)$.

Finally, we again remark that the same type of truncation techniques done in this subsection will also hold for the loss distributions derived in Section 4 and Section 5, and will be applied in all of the computations done in Section 8.

8. **Numerical examples when the default times are driven by a one-factor Gaussian copula model.** In the previous section, we studied the time-evolution of Value-at-Risk for a single stock over a two-year period in time steps of one month where the stock has jumps at default times driven by a CIR-process. In this section,

we will among other things study the time-evolution of Value-at-Risk for a portfolio of stocks over a 20 day period in time steps of one trading day, with jumps in all stock prices occurring at default times of an external group of defaultable entitles $\mathbf{C}_1, \ldots, \mathbf{C}_m$. Throughout this section, we assume that the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditionally independent, and are driven by a one-factor Gaussian copula model. First, in Subsection 8.1 we briefly discuss the model for the default times and present the parameters used in this framework. Then, we display related quantities such as the distribution of the number of defaults $\mathbb{P}\left[N_t^{(m)} = k\right]$, etc. Next, in Subsection 8.2 we study VaR for a portfolio consisting of $J = 70$ stocks by using the linear approximation formulas in Theorem 4.8. In Subsection 8.3, we consider a large portfolio with $J = 150$ stocks and then use the LPA (large portfolio approximation) formulas in Theorem 5.2 to compute VaR for this equity portfolio. Finally, in Subsection 8.4 we repeat similar studies as in Subsection 8.2, but now for a two-year period in steps of one month.

8.1. **The parameters and related quantities.** In this section, we assume that the stock prices $S_{t,j}$ for all companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ are given by Definition 4.1 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditionally independent, and driven by a one-factor copula model. Hence, the conditional default probability is the same for all entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$, and is given by

$$\mathbb{P}\left[\tau_i \leq t \,|\, Z\right] = \Phi\left(\frac{\Phi^{-1}\left(F(t)\right) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{8.1.1}$$

where Z is a standard normal random variable, ρ is the so-called default-correlation parameter, $\Phi(x)$ is the distribution function of a standard normal random variable. Furthermore, $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$ is the marginal default distribution same for all entities due to the exchangeability. For more about factor copula models, see, e.g., [40, 42, 47] or [29]. Furthermore, since the stock prices $S_{t,j}$ are given by Definition 4.1, the jumps $U_{n,j}$ in $S_{t,j}$ at the default times are i.i.d and exponentially distributed with parameter $\eta > 0$ the same for all companies $\mathbf{A}_j$.

In our numerical examples, we set $F(t) = \mathbb{P}\left[\tau_i \leq t\right] = 1 - e^{-\lambda t}$, which is standard in the credit literature, and calibrate λ so that the one-year default probability is same as in the CIR model in Section 7, that is, $0.0329 = 3.29\%$, and this gives $\lambda = 0.0335$, see Table 4. The motivation for having a one-year default probability of $0.0329 = 3.29\%$ is given in Subsection 7.1.

The "default-correlation" ρ in (8.1.1) is more challenging to estimate. Here, we simply set $\rho = 0.3$, that is, 30%, see Table 4, and in Subsection 8.3 we will also consider values of $\rho = 0.6 = 60\%$ to illustrate the effect of increasing default dependence leading to higher defaults and therefore more jumps in our the stock portfolio model. So, in this paper we only consider two values of the "unknown" default-correlation parameter ρ, however [25] investigated stock portfolio VaR as a function of ρ when it continuously runs through an interval on the positive real line bounded by one, where the stock prices $S_{t,j}$ are given by Definition 4.1 with default times $\tau_1, \tau_2 \ldots, \tau_m$ driven by a one-factor copula model in (8.1.1) with default-correlation parameter ρ.

In Table 7 on p. 60, we show among other things the expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ and $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ for $t = 1, 5, 10, 15, 20$ days when the individual default times are driven by a one-factor Gaussian copula model with parameters as

TABLE 4. The parameters and related quantities for the one-factor Gaussian copula model and the stock prices $S_{t,j}$ where we let $m = 125$.

Gauss copula	$m = 125$	$\rho = 0.3$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$		$\mathbb{P}[\tau_i \leq 1] = 0.0329 = 3.29\%$
$S_{t,j}$	$S_0 = 50$	$\mu = 0.15$	$\sigma = 0.2$	$\rho_S = 0.25$	$\eta = 21.98$	$\mathbb{E}[U_{n,j}] = \frac{1}{\eta} = 0.0455 = 4.55\%$

in Table 4 and where $m = 125$. From Table 7, we see that the expected number of defaults the first 20 days will never exceed one default, and consequently the expected number of jumps in the stock prices the first 20 days will also be less than one. By our assumption of exchangeability, we have that $\mathbb{E}\left[N_t^{(m)}\right] = m\mathbb{P}[\tau_i \leq t] = m(1 - e^{-\lambda t})$, so the individual default probabilities at $t = 1, 5, 10, 15, 20$ days are obtained from Table 7 by dividing the numbers for $\mathbb{E}\left[N_t^{(m)}\right]$ with m. Also note from Table 7, in the case with $\rho = 0.3$, we see that, after 10 days, there is a 0.1% probability of having 8 defaults or more among the entities in the exogenous group which are negatively affecting the stock prices in our equity portfolio, and after 15 days there is a 0.1% probability of 11 defaults or more among the entities in the same exogenous group.

Next, we turn to the parameters for the stock price model. First, note that the linearized loss distribution given in Theorem 4.8 will work for heterogeneous portfolios of arbitrary size J. However, for simplicity we will consider the homogeneous case, that is, the stock prices $S_{t,1}, \ldots, S_{t,J}$ satisfy (4.34) in Remark 4.9, so that $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$ and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$ in the stock portfolio. Furthermore, we let the parameters μ and σ be same as in the CIR-model case studied in Section 7 so that $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$, and the motivation for these values are given in Subsection 7.1. Regarding ρ_s, from Table 1 on p.369 in [44], we see that the average stock correlation in the period from 1963 to 2006 was 0.237, and [43] performs similar studies as in [44], but for the period from 1963 to 2022 and finds that the average stock correlation for this period is 0.264; see Table 1 in [43]. Inspired by the above observations, we set our stock correlation to $\rho_s = 0.25$, see Table 4. The jump parameter η is calibrated so that condition (7.1.4) will hold, that is, η is calibrated so that the defaults from the one-factor copula models "wipe" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. With the default and stock parameters as in Table 4, we then get that $\eta = 21.98$ via a numerical solver, so $\mathbb{E}[U_{n,j}] = \frac{1}{\eta} = 0.0455$, see Table 4.

With the one-factor Gaussian copula parameters in Table 4, we compute $\mathbb{P}\left[N_t^{(m)} = k\right]$ as described above, and in the left panel of Figure 4, for $m = 125$, the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, \ldots, 20$ days is shown. Furthermore, the right panel in Figure 4 displays the time evolution of the number of distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in normal scale where $k = 0, 1, \ldots, 18$ when $m = 125$ and $t = 1, 2, \ldots, 20$ days, where the default times have the same distribution as in Figure 4. The plots in Figure 4 were generated with the algorithms developed in [25], and in these figures we write t in days, but the actual computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$ are done with t measured in units of years.

So, for example, 2, 6, and 20 days mean that t is given by $t = \frac{2}{252}, \frac{6}{252}$, and $\frac{20}{252}$ in the formulas used for the computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$, where we remind that 252 is the average number of trading days on, e.g., the US-stock market.

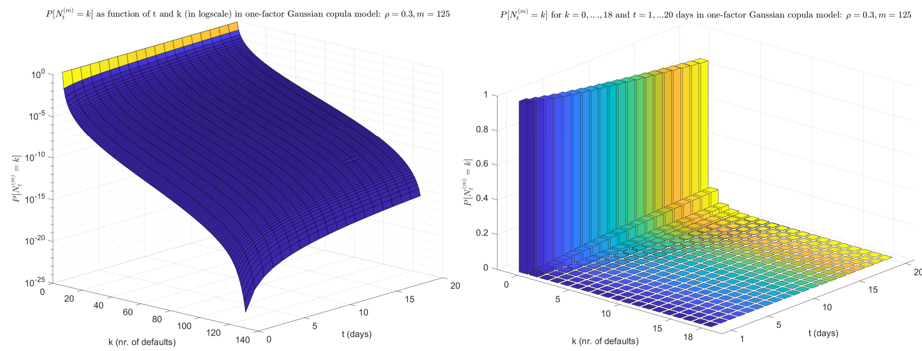

FIGURE 4. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $t = 1, 2, \ldots, 20$ days in a one-factor Gaussian copula model with parameters as in Table 4, where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, \ldots, 125$. **Right panel:** for $k = 0, \ldots, 18$.

8.2. **VaR over a 20-day period for a linearized portfolio of stocks when the jumps are due to default times driven by a one-factor Gaussian copula model.** In this subsection, we study Value-at-Risk for a portfolio of stocks as a function of time over a 20-day period in time steps of one trading day, with jumps in all stock prices occurring at default times $\tau_1, \tau_2 \ldots, \tau_m$, which are exchangeable, conditionally independent, and are driven by a one-factor copula model as discussed in Subsection 8.1 and with parameters as in Table 4. We study VaR for a portfolio of $J = 70$ stocks by using the linear approximation formulas in Theorem 4.8.

In Figure 5-6, we study the time evolution of Value-at-Risk (in % of V_0) for a portfolio of $J = 70$ stocks discussed in Subsection 8.1, where $t = 1, 2, \ldots, 20$ days, computed with same stock parameters as in Table 4. For $m = 125$, the left panel in Figure 5 displays the time evolution of Value-at-Risk in % of V_0 for $t = 1, 2, \ldots, 20$ days in the case when S_t has jumps coming from default times in a one-factor Gaussian copula model with parameters as in Table 4. The right panel in Figure 5 displays the Black-Scholes case for the stock price, i.e. with no jumps in S_t, which has the same drift and volatility parameters as in the left panel. From the left panel of Figure 5, looking at the red line (99.9%-VaR), we see that, for $t = \frac{12}{252}$, that after 12 days there is a 0.1% probability of having a loss in the stock portfolio of 42% or bigger, of the initial portfolio value V_0 at time $t = 0$. Furthermore, in Figure 6 we plot the time evolution of the relative difference of Value-at-Risk (in %) between the case with jumps in the stock prices $S_{t,j}$ coming from default times in a one-factor Gaussian copula model with parameters as in Table 4, and the standard Black-Scholes case, i.e. without jumps. The relative difference is measured with respect to the Black-Scholes case. The rest of the parameters for $S_{t,j}$ are the same as in Table 4. As can be seen in Figure 6, introducing downward jumps in $S_{t,j}$ at the default times $\tau_1, \tau_2 \dots, \tau_m$, which comes from a one-factor Gaussian copula model, will for example increase the 99.9% VaR up to around 1450% compared to the Black-Scholes model, and for the 99% VaR up to 765%. Furthermore, we also note the curves in the left panel of Figure 5 are not as smooth as in the left panel of Figure 2, and the reason for this discontinuity will be explained in Subsection 8.3. All the VaR computations in the left panel of Figure 5 are done by numerically solving the equation $F_{L_t^{\Delta V}}(x) = \alpha$ where $F_{L_t^{\Delta V}}(x) = \mathbb{P}\left[L_t^{\Delta V} \leq x\right]$ is computed using Theorem 4.8 under condition (4.34) in Remark 4.9 so that the mappings $\Psi_k^V(x, t, \mu, \sigma, S_0, \rho_S, \eta)$ in $F_{L_t^{\Delta V}}(x)$ are given by (4.35)-(4.36). Furthermore, in our computations of $F_{L_t^{\Delta V}}(x)$, we use the same truncation techniques as discussed in Subsection 7.3. Finally, the VaR computations in the right panel of Figure 5 are done using Equation (4.43) in Corollary 4.11 for the "Black-Scholes" linear portfolio case.

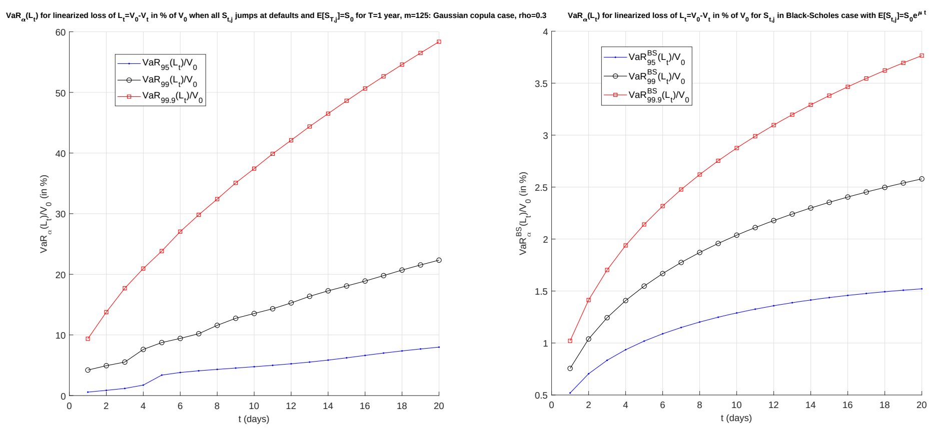

FIGURE 5. $m = 125$: The time evolution of Value-at-Risk (in % of V_0) of a linearized stock portfolio with $J = 70$ stocks for $t = 1, 2, \dots, 20$ days. **Left panel:** The case with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4. **Right panel:** The Black-Scholes case, i.e. without jumps, and where the drift and volatility are same as in the left panel.

As discussed in Section 5, the linearized loss $L_t^{\Delta V}$ will only work somewhat accurately as an approximation of the true loss $L_t^{(V)}$ when the time t is small, that is, if $|X_{t,j}|$ is small for all j when t is small. Recall that $X_{t,j}$ is defined as in (4.11), and from the expression in (4.11), it is clear that the more potential number of jump terms $\sum_{n=1}^{N_t^{(m)}} U_{n,j}$ in the expression for $X_{t,j}$, the more defaultable entities m, and the less likely it will be that $|X_{t,j}|$ is "small". Thus, $|X_{t,j}|$ should in general grow in the number of defaultable entities m. So, it is therefore of interest to study $S_{t,j}$ and its linear approximation $S_0(1 + X_{t,j})$ as a function of the number of defaulted entities m for different time points t. Hence, Figure 7 displays the expected value of $S_{t,j}$ and its linear approximation $S_0(1 + X_{t,j})$, that is, $\mathbb{E}[S_{t,j}]$ and $S_0\mathbb{E}[(1 + X_{t,j})]$

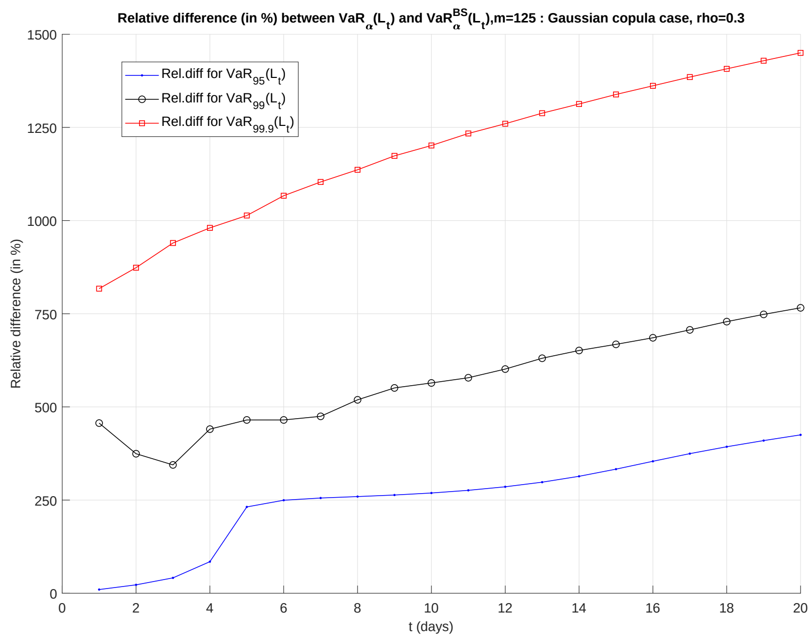

FIGURE 6. The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between the case of linearized stock portfolio with $J = 70$ stocks with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4, and the linearized Black-Scholes case, i.e. without jumps, where drift and volatility is same as in the jump case. The relative difference is measured with respect to the Black-Scholes case.

as a function of the number of defaulted entities m for $t = 5, 10, 20$ and $t = 252$ days where $X_{t,j}$ is defined as in Equation (4.11) with parameters as in Table 4. The jumps in the stock price occur at default times driven by a one-factor Gaussian copula model with parameters as in Table 4. The number of defaultable entities m runs from 5 up to 135 in Figure 7.

Furthermore, Figure 8 shows the relative difference between $\mathbb{E}[S_{t,j}]$ and $S_0\mathbb{E}[(1 + X_{t,j})]$ in percent, as a function of the number of defaulted entities m for $t = 5, 10, 20$ days in the left panel and for $t = 252$ days in the right panel, where $X_{t,j}$ is defined as in Equation (4.11), with the model and parameters the same as in Figure 7. The relative difference is measured with respect to $\mathbb{E}[S_{t,j}]$. From Figure 8, we see that the relative error, or difference, for $t = 5$ days never exceeds 0.07% when $m \leq 135$. Also, when $t = 1$ year, that is, $t = 252$ days, then the relative error is always smaller than 6%. In all plots in Figure 8, the relative difference is increasing when $m \geq 20$.

8.3. **VaR over a 20-day period for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a one-factor Gaussian copula model.** In this subsection, we study Value-at-Risk for a large

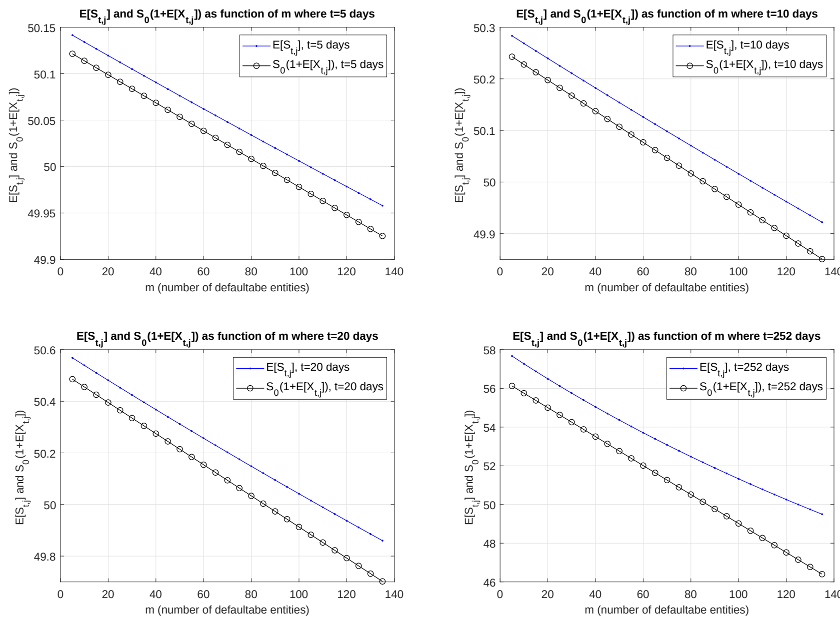

FIGURE 7. Expected value of $S_{t,j}$ and its linear approximation $S_0\left(1 + X_{t,j}\right)$, that is, $\mathbb{E}\left[S_{t,j}\right]$ and $S_0\mathbb{E}\left[(1 + X_{t,j})\right]$ as a function of the number of defaulted entities m for $t = 5, 10, 20$, and 252 days where $X_{t,j}$ is defined as in Equation (4.11) with parameters as in Table 4. The jumps in the stock price occur at default times driven by a one-factor Gaussian copula model with parameters as in Table 4.

homogeneous portfolio of stocks as a function of time over a 20-day period in time steps of one trading day. The stock prices in the portfolio have jumps occurring at default times $\tau_1, \tau_2 \ldots, \tau_m$ which are exchangeable, conditionally independent, and are driven by a one-factor Gaussian copula model as discussed in Subsection 8.1 and with parameters as in Table 4. We study VaR for a portfolio of $J = 150$ stocks by using the LPA approximation formulas in Theorem 5.2, and we do our VaR studies for two different levels of the default correlation parameter ρ in the one-factor Gaussian copula model. First, in the left panel of Figure 9 we display the time evolution of Value-at-Risk in % of V_0 for $t = 1, 2, \ldots, 20$ days in the case when $S_{t,j}$ has jumps coming from default times in a one-factor Gaussian copula model with parameters as in Table 4, so the default-correlation ρ is set to $\rho = 0.3$. The right panel in Figure 9 displays the same quantities as in the left panel, but now with the default-correlation parameter $\rho = 0.6$ and $\eta = 13.92$ so that condition (7.1.4) holds, just as in the left panel of Figure 9. Comparing the VaR-curves in the left and right panel in Figure 9, we see that the 99% and 99.9% VaR plots for $\rho = 0.6$ in the right panel are much higher than the corresponding curves for $\rho = 0.3$ in the left panel where $\eta = 13.92$, with the rest of the parameters the same as in the left panel. For example, looking at the red line (99.9% VaR) in the right panel

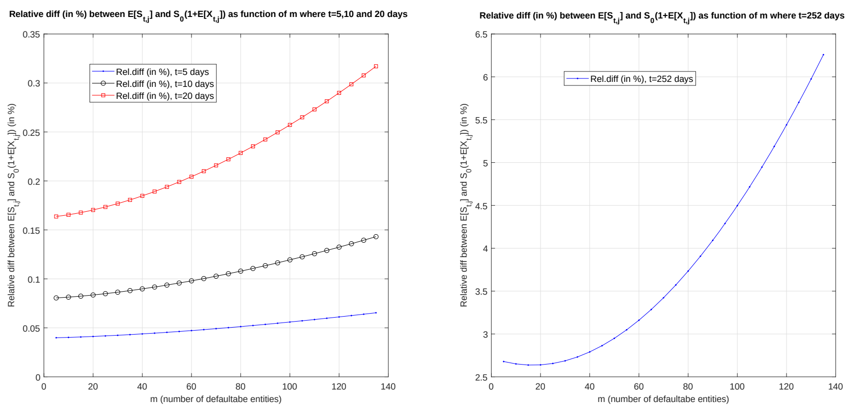

FIGURE 8. Relative difference between $\mathbb{E}\left[S_{t,j}\right]$ and $S_0\mathbb{E}\left[(1+X_{t,j})\right]$ in percent as a function of the number of defaulted entities m for different time points t where $X_{t,j}$ is defined as in Equation (4.11) with model and parameters the same as in Figure 7. The relative difference is measured with respect to $\mathbb{E}\left[S_{t,j}\right]$. **Left panel:** for $t = 5, 10, 20$ days. **Right panel:** for $t = 252$ days.

with $\rho = 0.6$, we see that after 12 days there is a 0.1% probability of having a loss in the portfolio of 80% or more than the initial portfolio value V_0 at time $t = 0$. However, when $\rho = 0.3$ in the left panel, there is for the same time, that is, 12 days, a 0.1% probability of having a loss in the portfolio of 33% or more than the initial portfolio value V_0 at time $t = 0$. The big differences between the curves for the same α-levels in the two panels are due to the fact that a default-correlation of $\rho = 0.6$ will create probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ that are substantially larger for lower k-values compared to the corresponding probabilities in the case when $\rho = 0.3$. Looking at the left panel in Figure 11, which displays the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ on the log-scale for $k = 0, ..., 125$ and $t = 1, 2, \ldots, 20$ days in a one-factor Gaussian copula model where $\rho = 0.6$, and comparing these probabilities with the corresponding values for $\mathbb{P}\left[N_t^{(m)} = k\right]$ in the left panel of Figure 4 where $\rho = 0.3$, we see that the levels of $\mathbb{P}\left[N_t^{(m)} = k\right]$ when $\rho = 0.6$ for some k are a factor 10^5 higher compared with the probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ when $\rho = 0.3$ for the same k-values.

Furthermore, we also note that the curves in both of panels of Figure 9 display a non-smooth behavior. The main reason for the somewhat discontinuous behavior of the graphs in Figure 9 are explained by looking at the middle and right panels in Figure 19 on p.66, which display the time evolution of Value-at-Risk at $t = 1, 2, \ldots, 20$ for the default counting process $N_t^{(m)}$ driven by a one-factor Gaussian copula model with $m = 125$, parameters as in Table 4, and default-correlation parameter $\rho = 30\%$ (middle panel) and $\rho = 60\%$ (right panel). Recall that $N_t^{(m)}$ is a counting process, so the curves in Figure 19 will be piecewise constant and increasing. Comparing the left and right panels in Figure 9 with the middle and right panels in Figure 19, we clearly see that the discontinuities, i.e. "jumps", at

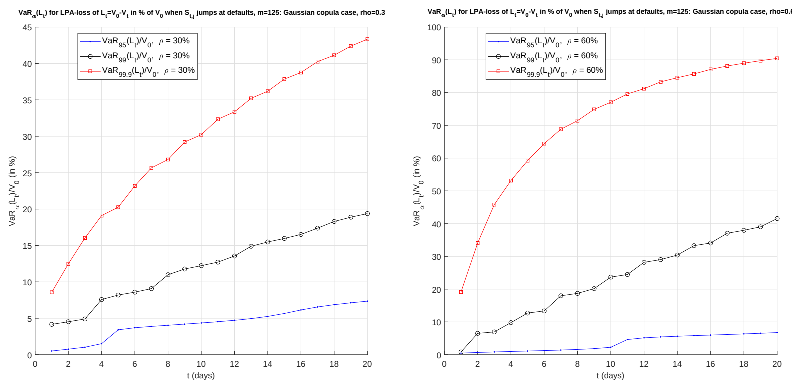

FIGURE 9. The time evolution of Value-at-Risk (in % of V_0) computed with the LPA-formula in Theorem 5.2 for $t = 1, 2, \ldots, 20$ days of a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4. **Left panel:** Default-correlation parameter $\rho = 30\%$ and $\eta = 21.98$. **Right panel:** Default-correlation parameter $\rho = 60\%$ and $\eta = 13.92$. In both panels, condition (7.1.4) holds.

different time points in the middle and right panels of Figure 19 coincide in time with the somewhat discontinuous behavior of the graphs in the left and right panels of Figure 9. The main reason for the similar discontinuous behavior in Figure 9 and the middle-right panels in Figure 19 is that the computations are done over a very short time period of 20 days, in steps of one trading day, leading to a quite degenerated distribution for $\mathbb{P}\left[N_t^{(m)} = k\right]$ over k as seen in the right panel of Figure 4. More specifically, for $t = 1, 2, \ldots, 20$ days, the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ will have a very high probability for $k = 0$ ("no defaults") almost equal to one, while $\mathbb{P}\left[N_t^{(m)} = k\right]$ will be very small for $k \geq 1$. Furthermore, the distribution function $F_{N_t^{(m)}}(x)$ for $N_t^{(m)}$ will for small time points therefore have similar behavior to the LPA-distribution $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ given by (5.5), or, equivalently, (5.24). Hence, for fixed t, the tail behavior of $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$ and $F_{N_t^{(m)}}(x)$ will display similar characteristics for smaller time points, explaining the discontinuous behavior of the graphs in Figure 9, particulary when comparing with the middle and right panels in Figure 19. If $\mathbb{P}\left[N_t^{(m)} = k\right]$ is computed over a long period, such as two years, then $\mathbb{P}\left[N_t^{(m)} = k\right]$ will have quite large probabilities also for $k \geq 1$, see e.g. the right panel in Figure 13. Hence, as time t increases (say, one year or more), the distribution of $\mathbb{P}\left[N_t^{(m)} = k\right]$ over k will be less "degenerated" leading to a more smooth curve for the tail behavior of $F^{\mathrm{LPA}}_{L_t^{(V)}}(x)$, and therefore more smooth VaR curves for the stock portfolios, see for example in Subsection 8.4, for longer periods, such as two years, will lead to very smooth VaR-curves in the Gaussian one-factor case, with the same parameters as in Table 4.

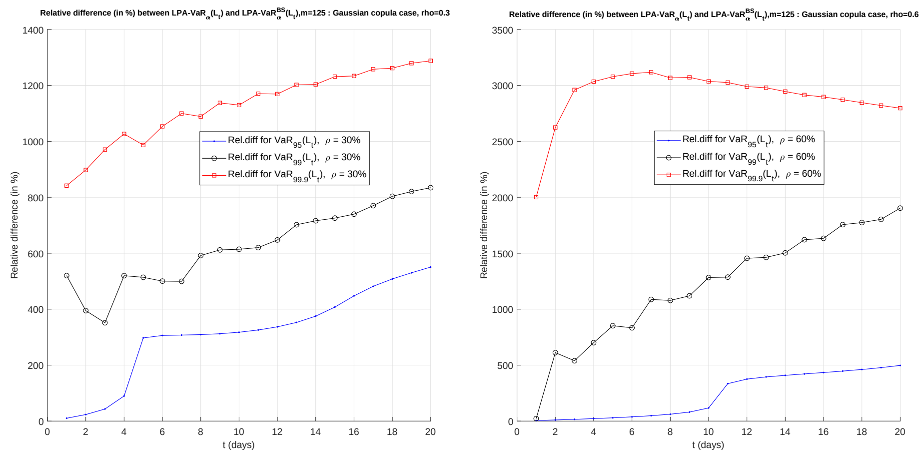

FIGURE 10. The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between a stock portfolio with jumps as in Figure 9 using the LPA formula in Theorem 5.2 and the standard Black-Scholes case, i.e. without jumps, given by the right panel in Figure 11. The relative difference is measured with respect to the Black-Scholes case. All parameters for the jump-model are as in Figure 9. **Left panel:** With default-correlation parameter $\rho = 30\%$ and $\eta = 21.98$. **Right panel:** With default-correlation parameter $\rho = 60\%$ and $\eta = 13.92$.

Next, in the two panels in Figure 10, we display the time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between a stock portfolio with jumps as in Figure 9 using the LPA-formula (5.5) in Theorem 5.2 and the standard Black-Scholes case, i.e. without jumps, given in the right panel of Figure 11 computed with the Black-Scholes LPA-formula in Equation (5.30) with parameters as in Table 4. As can be seen in Figure 10, the differences between the jump vs, non-jump VaR-cases are huge. For example, the 99.9% VaR for $\rho = 0.6$ in the right panel is for some time points around 3000% higher than the corresponding 99.9% VaR values in the Black-Scholes portfolio case. In our VaR-computations in Figure 9, we use the same truncation techniques for the LPA-portfolio loss distributions as discussed in Subsection 7.3.

8.4. **VaR over a 2-year period for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a one-factor Gaussian copula model.** In this subsection, we repeat similar studies for the same model and same parameters as in Subsection 8.2, but now for a two-year period in steps of one month. The obtained VaR-curves in this subsection will be smooth and continuous, just as in the CIR-case where we also studied VaR over a two-year period. Hence, Figure 12 shows the same type of VaR-curves as in Figure 5, but for a two-year period, and all parameters in Figure 12 are the same as in Figure 5, and given by Table 4. By comparing the curves in the left panel of Figure 12 with the graphs in left panel of Figure 5, we clearly see that the VaR values over a two-year period are very smooth and continuous. Unsurprisingly, the VaR

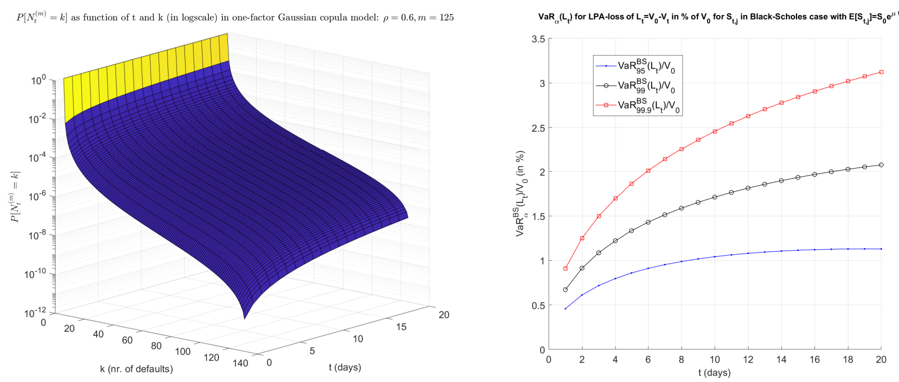

FIGURE 11. **Left panel:** The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale for $k = 0, ..., 125$ and $t = 1, 2, \ldots, 20$ days in a one-factor Gaussian copula model where $m = 125, \rho = 0.6, \eta = 13.92$, and the rest of the parameters are the same as in Table 4. **Right panel:** The time evolution of Value-at-Risk (in % of V_0) for $t = 1, 2, \ldots, 20$ days of a homogeneous portfolio with $J = 150$ stocks in the Black-Scholes case computed with the LPA-formula in Equation (5.30) and with parameters as in Table 4.

values for the two-year period are also much higher than for the 20-day period. For example, looking at the red line (99.9% VaR) in Figure 12, we see that, after 12 months, there is a 0.1% probability of having a loss in the portfolio which is 90% or more than the initial portfolio value V_0 at time $t = 0$.

In Figure 13, we display the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $t = 1, 2, \ldots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 4 where $m = 125$ and $\rho = 0.3$. Comparing the probabilities in Figure 13 over a two-year period with those in Figure 4 over a 20-day period, we see that there are huge differences. Furthermore, in the two-year case our probabilities are now much less degenerated, i.e. not centered around $k = 0$, as in the 20-day period, and this fact also explains the much more smooth curves in in the left panel of Figure 12 compared with those in left panel of Figure 5. All computations in Figure 12 are done as in Subsection 8.3 and with the same parameters, and the only difference is that we now consider a two-year period in steps of one month. Furthermore, just as in previous subsections, we will in our VaR computations in Figure 12 use the same truncation techniques for the LPA portfolio loss distributions as discussed in Subsection 7.3.

Note that the right panel in Figure 12 shows the VaR-values for the Black-Scholes case, i.e. without jumps, using the LPA-formula in Equation (5.30) and with the same drift, stock-correlation, and volatility parameters as in the left panel, see Table 4. From the right panel in Figure 12, we see that in the Black-Scholes LPA portfolio model it is extremely difficult to obtain losses over a two-year period, where we remind that a negative loss is a gain. For example, we see that after 20 months there is a 95% probability of having a gain which is 15.2% or more of the initial portfolio value V_0. Similarly, after 20 months there is a 99% probability of

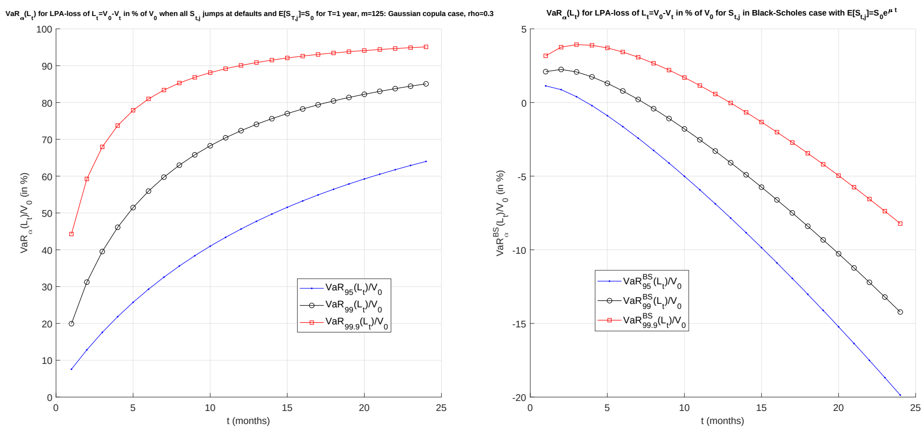

FIGURE 12. The time evolution of Value-at-Risk (in % of V_0) computed with the LPA-formulas in a homogeneous portfolio with $J = 150$ stocks for $t = 1, 2, \ldots, 24$ months. **Left panel:** The case with jumps in the stock price where the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and using the LPA-formula in Theorem 5.2. **Right panel:** The Black-Scholes case, i.e. without jumps, using the LPA-formula in Equation (5.30) and with parameters as in Table 4.

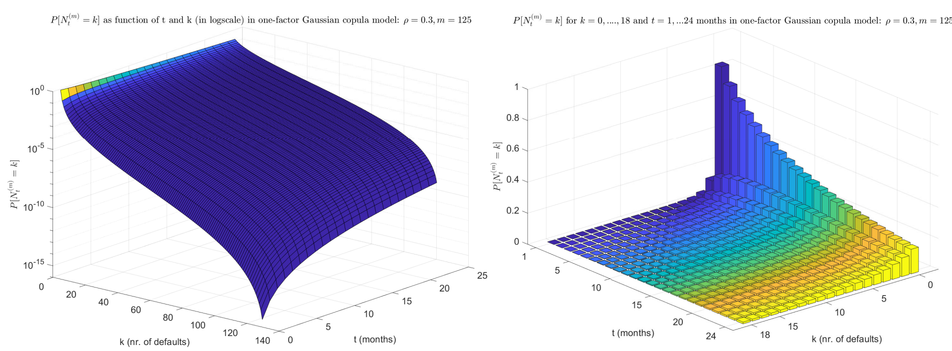

FIGURE 13. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $t = 1, 2, \ldots, 24$ months in a one-factor Gaussian copula model with parameters as in Table 4 where $m = 125$ and $\rho = 0.3$. **Left panel:** in log-scale for $k = 0, ..., 125$. **Right panel:** for $k = 0, ..., 18$. The plots in the panels are viewed from different angles.

having a gain which is 10.3% or more of the initial portfolio value V_0, and 99.9% probability of having a gain which is 4.96% or more of the initial portfolio value V_0. The intuitive explanation of these VaR results in the Black-Scholes LPA portfolio setting is that the growth rate will for longer time periods beat the downside risk given by the volatility term, while such positive stock prognoses are never possible in the corresponding stock price model with jumps at external defaults over the same time period of 20 months, as clearly seen in the left panel of Figure 12.

TABLE 5. The expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ and $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ for $t = 1, 6, 12, 18, 24$ months when individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$.

t (in months)	1	6	12	18	24
$\mathbb{E}\left[N_t^{(m)}\right]$	0.3480	2.073	4.113	6.118	8.090
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$	13	39	55	66	74

In Table 5, we show the expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ for $t = 1, 3, 6, 12,$ $18, 24$ months when the individual default times are driven by a one-factor Gaussian copula model with parameters as in Table 4 and where $m = 125$. So, from Table 5 we see that our one-factor Gaussian copula model implies that we expect, for example, around 2 defaults in six months, 4 defaults in one year, and 8 defaults in two-years. Consequently, this is also the number of jumps that we expect to occur in our stock price up to each of these time points, where each jump has the expected size of $\mathbb{E}[U] = \frac{1}{\eta}$. By our assumption of exchangeability, we have that $\mathbb{E}\left[N_t^{(m)}\right] = m\mathbb{P}[\tau_i \leq t]$, so the individual default probabilities at $t = 1, 3, 6, 12, 18, 24$ months are obtained from Table 5, by dividing the numbers for $\mathbb{E}\left[N_t^{(m)}\right]$ with m. From Table 5 we also see that after 6 months there is a 0.1% probability of having 39 defaults or more among the entities in the exogenous group, which are negatively affecting the stock prices in our equity portfolio, and after 24 months (i.e. 2 years) there is a 0.1% probability of 74 defaults or more among the entities in the exogenous group, negatively affecting the stock prices in our equity portfolio, when using the parameters in Table 4.

9. **Numerical examples when the default times are driven by a Clayton copula model.** In the previous section, we studied the time-evolution of Value-at-Risk for stock portfolios where the stock prices have jumps at default times driven by a one-factor Gaussian copula model.

In order to aim for generality and robustness checking, it is of interest to also study stock portfolio VaR values when the default times $\tau_1, \tau_2 \dots, \tau_m$ are generated by copulas other than the one-factor Gaussian copula model. Therefore, in this section we perform similar studies as in Subsection 8.3, but now for a stock price model where the default times are exchangeable, conditionally independent, and are driven by a Clayton copula model. First, in Subsection 9.1 we briefly discuss the model for the default times and present the parameters used in this framework. The marginal default distribution $F(t) = \mathbb{P}[\tau_i \leq t]$ will in this section be the same as in Section 8 for the one-factor Gaussian copula model. Furthermore, since we want to compute the VaR-values under similar conditions as in the one-factor Gaussian copula model in Subsection 8.3, we will choose the parameters in the Clayton copula model so that the one-year default correlation is the same in both models. With the parameters fixed in Subsection 9.1, we also display related quantities, such as the distribution of the number of defaults $\mathbb{P}\left[N_t^{(m)} = k\right]$, etc.

Then, in Subsection 9.2 we consider a large portfolio with $J = 150$ stocks and then use the LPA (large portfolio approximation) formulas in Theorem 5.2 to compute VaR for this equity portfolio with parameters as in Subsection 9.1. We also compare the stock portfolio VaR values in the Clayton copula case both with the Black-Scholes case (no jumps) and when the default times come from a one-factor Gaussian copula model.

9.1. **The parameters and related quantities.** In this section, we assume that the stock prices $S_{t,j}$ for all companies $\mathbf{A}_1, \ldots, \mathbf{A}_J$ are given by Definition 4.1 where $N_t^{(m)} = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$ and the default times $\tau_1, \tau_2 \ldots, \tau_m$ to the entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable, conditionally independent, and driven by a Clayton copula model. Hence, the conditional default probability is the same for all entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ and is given by

$$\mathbb{P}\left[\tau_i \leq t \,|\, Z\right] = \exp\left(Z\left(1 - F(t)^{-\theta}\right)\right) \tag{9.1.1}$$

where Z is a gamma-distributed random variable with parameters $a = \frac{1}{\theta}$ and $b = 1$ so that its density $f_Z(z)$ is given by $f_Z(z) = \frac{z^{\frac{1-\theta}{\theta}} e^{-z}}{\Gamma\left(\frac{1}{\theta}\right)}$ for $z \geq 0$. Furthermore, $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$ is the marginal default distribution and is the same for all entities due to the exchangeability. By using the fact that the Clayton copula belongs to the family of Archimedean copulas, it is straightforward to prove that the default correlation $\mathrm{Corr}\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right)$ in a homogeneous group of obligors is given by

$$\mathrm{Corr}\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right) = \frac{\left(\frac{2}{F(t)^{\theta} - 1}\right)^{-\frac{1}{\theta}} - F(t)^2}{F(t)\left(1 - F(t)\right)} \tag{9.1.2}$$

and this relation can be used to benchmark the Clayton copula against other factor models, as will be seen below. For more about the Clayton copula model, see [8,29] or [40]. Furthermore, since the stock prices $S_{t,j}$ are given by Definition 4.1, the jumps $U_{n,j}$ in $S_{t,j}$ at the default times are i.i.d and exponentially distributed with the parameter $\eta > 0$ the same for all companies $\mathbf{A}_j$.

Since we want to compute the VaR-values under similar conditions as in the one-factor Gaussian copula model in Subsection 8.3, we will use the same marginal default distribution $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$ as in Section 8 for the one-factor Gaussian copula model, see Table 6. Furthermore, given $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$, we choose the Clayton-copula parameter θ so that the one-year default correlation is the same as in the one-factor Gaussian copula model used in Subsection 8, where VaR values for two different correlation parameters ρ were studied, $\rho = 0.3 = 30\%$ and $\rho = 0.6 = 60\%$. While the Clayton copula model allows for an explicit expression of the default correlation $\mathrm{Corr}\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right)$ as stated in (9.1.2), there is no explicit formula for $\mathrm{Corr}\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right)$ in the one-factor Gaussian copula model. However, it is still easy to numerically compute $\mathrm{Corr}\left(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}}\right)$ in the one-factor Gaussian copula model, which for $t = 1$ and $\rho = 30\%$ gives $\mathrm{Corr}\left(1_{\{\tau_i \leq 1\}}, 1_{\{\tau_j \leq 1\}}\right) = 0.0812$, and for $\rho = 60\%$ with $t = 1$ we get $\mathrm{Corr}\left(1_{\{\tau_i \leq 1\}}, 1_{\{\tau_j \leq 1\}}\right) = 0.2467$. Hence, in this section we use two different parameters θ corresponding to the two ρ-parameters ($\rho = 30\%$ and $\rho = 60\%$) in Subsection 8.3 so that these θ's render the same values for the one-year default correlation $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right)$ when using the formula in (9.1.2) with $t = 1$. Thus, numerically solving for θ in (9.1.2) with the numerical values for the one-year default correlation coming from the one-factor Gaussian copula model yields $\theta = 0.169$ in the Clayton copula case when $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right) = 0.0812$, i.e. $\rho = 30\%$ in the Gaussian copula case, and $\theta = 0.44$ when $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right) = 0.2467$, i.e. $\rho = 60\%$ in the one-factor Gaussian copula model, see Table 6.

TABLE 6. The parameters and related quantities for the Clayton copula model and the stock prices $S_{t,j}$ where we let $m = 125$.

Clayton copula	$m = 125$	$\theta = 0.169, \theta = 0.44$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$	$\mathbb{P}[\tau_i \leq 1] = 0.0329$
$S_{t,j}$	$S_0 = 50$ $\mu = 0.15$	$\sigma = 0.2$ $\rho_S = 0.25$	$\eta = 21.67\,(\theta = 0.169)$	$\eta = 13.48\,(\theta = 0.44)$	

With the Clayton copula parameters in Table 6, we compute $\mathbb{P}\left[N_t^{(m)} = k\right]$, and Figure 14 plots, for $m = 125$, the time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale where $k = 0, ..., 125$ and $t = 1, 2, \ldots, 20$ days. The left panel in Figure 14 displays $\mathbb{P}\left[N_t^{(m)} = k\right]$ for the case $\theta = 0.169$, while the right panel in Figure 14 shows $\mathbb{P}\left[N_t^{(m)} = k\right]$ for $\theta = 0.44$. The plots in Figure 14 were generated with the algorithms developed in [25], and in these figures we write t in days, but the actual computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$ are done with t measured in units of years. So, for example, 2, 6 and 20 days mean that t is given by $t = \frac{2}{252}, \frac{6}{252}$, and $\frac{20}{252}$ in the formulas used for the computations of $\mathbb{P}\left[N_t^{(m)} = k\right]$, where we remind that 252 is the average number of trading days on the US stock market. By comparing the left panel in Figure 14 with the left panel in Figure 4 and the right panel in Figure 14 with the left panel in Figure 11, we clearly see that the Clayton copula model consistently creates higher probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ compared with the one-factor Gaussian copula model, even when both copula-models have identical marginal default distributions and the same one-year default correlation $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right)$ in both model comparisons, that is in the comparison $\theta = 0.169$ (Clayton) vs. $\rho = 0.3$ (Gaussian) and the comparison $\theta = 0.44$ (Clayton) vs. $\rho = 0.6$ (Gaussian). To further quantify the large differences in probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ coming from the Gaussian and Clayton copula, we display $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ in Table 7 for $\alpha = 99.9\%$ in both the Gaussian and Clayton copula for $\theta = 0.169, \theta = 0.44$ (Clayton), and $\rho = 0.3, \rho = 0.6$ (Gaussian). From Table 7, we clearly see that the tail probabilities in the Clayton copula case are much higher than in the one-factor Gaussian copula framework with the same default probabilities and same one-year default correlation $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right)$. For example, after 5 days $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ is twice as big as when $\theta = 0.169$ in the Clayton copula compared to the Gaussian copula with $\rho = 0.3$ (giving the same one-year default correlation $\mathrm{Corr}\left(1_{\{\tau_i\leq 1\}}, 1_{\{\tau_j\leq 1\}}\right) = 0.0812$). After 20 days, there are $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right) = 21$ defaults in the Clayton model compared with

$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right) = 13$ for Gaussian case (with same default probabilities and same one-year default correlation $\mathrm{Corr}\left(1_{\{\tau_i \le 1\}}, 1_{\{\tau_j \le 1\}}\right) = 0.2467$).

The full time evolution at $t = 1, 2, \ldots, 20$ of $\mathrm{VaR}_\alpha\left(N_t^{(m)}\right)$ for $\alpha = 95\%, 99\%$, and 99.9% are displayed in Figure 17 for $N_t^{(m)}$ driven by a Clayton copula, and Figure 19 when $N_t^{(m)}$ is generated by a one-factor Gaussian copula.

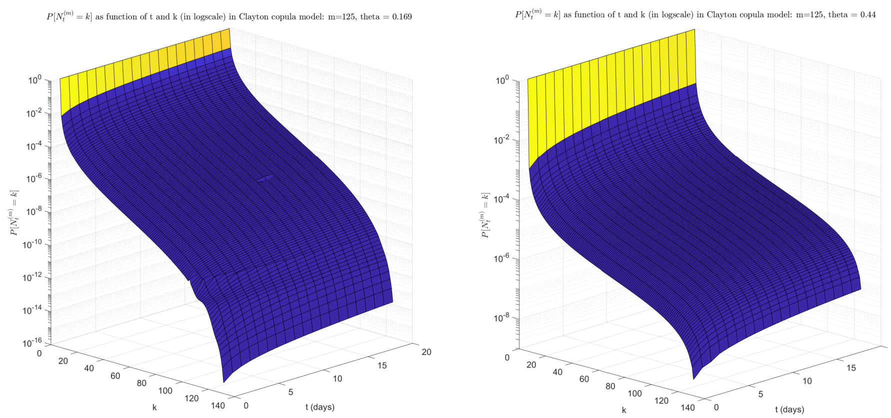

FIGURE 14. The time evolution of the distribution $\mathbb{P}\left[N_t^{(m)} = k\right]$ in log-scale for $t = 1, 2, \ldots, 20$ days and $k = 0, ..., 125$ in a Clayton copula model with parameters as in Table 6 where $m = 125$. **Left panel:** For $\theta = 0.169$. **Right panel:** For $\theta = 0.44$.

Next, we turn to the parameters for the stock price model. Since in this section we will only consider a homogeneous stock portfolio, all stock prices $S_{t,1}, \ldots, S_{t,J}$ satisfy (4.34) in Remark 4.9, so that $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$, and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$ in the stock portfolio. Furthermore, we let the parameters μ and σ be same as in the CIR-model case studied in Section 7 and the one-factor Gaussian copula model in Section 8 so that $S_0 = 50, \mu = 0.15 = 15\%$, and $\sigma = 0.2 = 20\%$, and the motivation for these values are given in Subsection 7.1, see Table 6. We let the stock correlation parameter ρ_S be $\rho_S = 0.25$ with the same motivation as given in Subsection 8.1, see in Table 6. The jump parameter η is calibrated so that condition (7.1.4) will hold, that is, η is calibrated so that the defaults from the Clayton copula models "wipe" out the expected one-year log-growth for a corresponding Black-Scholes model with drift $\mu = 15\%$ and where $m = 125$. This is the same assumption as in Section 8, and will make our stock portfolio VaR comparisons between the Clayton and Gaussian copula default models be somewhat "fair". With the default and stock parameters as in Table 6, we then get for the case $\theta = 0.169$ that $\eta = 21.67$ via a numerical solver so $\mathbb{E}\left[U_{n,j}\right] = \frac{1}{\eta} = 0.0462$, while $\theta = 0.44$ implies that $\eta = 13.478$ so that $\mathbb{E}\left[U_{n,j}\right] = \frac{1}{\eta} = 0.0742$, see Table 6.

9.2. **VaR over a 20-day period for a large homogeneous stock portfolio where jumps in stocks are due to default times driven by a Clayton copula model.** In this subsection, we study Value-at-Risk for a large homogeneous portfolio of stocks as function of time over a 20-day period in time steps of one

Table 7. The expected number of defaults $\mathbb{E}\left[N_t^{(m)}\right]$ and $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ for $t = 1, 5, 10, 15, 20$ days when individual default times are driven by a one-factor Gaussian copula and a Clayton copula model with parameters as in Table 4 and Table 6 where $m = 125$. Both copula models have the same marginal default distributions and same one-year default correlations for each pair $\rho = 0.3$ vs. $\theta = 0.169$ and $\rho = 0.6$ vs. $\theta = 0.44$.

t (in days)	1	5	10	15	20
$\mathbb{E}\left[N_t^{(m)}\right]$	0.0166	0.0829	0.1658	0.2487	0.3314
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ Gaussian, $\rho = 0.3$	2	5	8	11	13
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ Clayton, $\theta = 0.169$	3	10	14	18	21
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ Gaussian, $\rho = 0.6$	3	13	21	28	34
$\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ Clayton, $\theta = 0.44$	3	21	34	43	49

trading day. The stock prices in the portfolio have jumps occurring at default times $\tau_1, \tau_2 \ldots, \tau_m$ which are exchangeable, conditionally independent, and are driven by a Clayton copula model as discussed in Subsection 9.1 and with parameters as in Table 6. We study VaR for a portfolio of $J = 150$ stocks by using the LPA approximation formulas in Theorem 5.2, and we do our VaR studies for two different levels of the parameter θ in the Clayton copula model. First, in the left panel of Figure 15 we display the time evolution of Value-at-Risk in % of V_0 for $t = 1, 2, \ldots, 20$ days in the case when $S_{t,j}$ has jumps coming from default times in a Clayton copula model with parameters as in Table 6, where the θ-parameter is set to $\theta = 0.169$ and $\eta = 21.67$. The right panel in Figure 15 displays the same quantities as in the left panel, but now with Clayton parameter $\theta = 0.44$ and where the stock-jump parameter is $\eta = 13.48$ so that condition (7.1.4) holds, just as in the left panel of Figure 15. Comparing the VaR-curves in the left and right panels in Figure 15, we see that the 99% and 99.9% VaR plots for $\theta = 0.44$ in the right panel are much higher than the corresponding curves for $\theta = 0.169$ in the left panel where $\eta = 13.48$, with the rest of the parameters the same as in the left panel. For example, looking at the red line (99.9% VaR) in the right panel with $\theta = 0.44$, we see that after 20 days there is a 0.1% probability of having a loss in the portfolio which is 97% or more than the initial portfolio value V_0 at time $t = 0$. However, when $\theta = 0.169$ in the left panel, there is for the same time, that is, 20 days, a 0.1% probability of having a loss in the portfolio which is 59.9% or more than the initial portfolio value V_0 at time $t = 0$. The big differences between the curves for the same α-levels in the two panels are due to the fact that a Clayton parameter $\theta = 0.44$ will create probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ that are substantially larger for lower k-values compared to the corresponding probabilities in the case when $\theta = 0.169$, which is clearly seen when comparing the left and right panels in Figure 14. From Figure 15, we also observe that the probabilities $\mathbb{P}\left[N_t^{(m)} = k\right]$ in the Clayton copula model are higher than in the one-factor Gaussian copula model displayed in the left panel of Figure 4 with $\rho = 0.3$ and the left subplot of Figure 11 with $\rho = 0.6$, with same marginal default distributions and the same one-year default correlation Corr $\left(1_{\{\tau_i \le 1\}}, 1_{\{\tau_j \le 1\}}\right)$ as in the Clayton copula model ($\theta = 0.169$ vs. $\rho = 0.6$ and $\theta = 0.44$ vs. $\rho = 0.3$), and this explains the higher VaR values in Figure 15 compared to Figure 9.

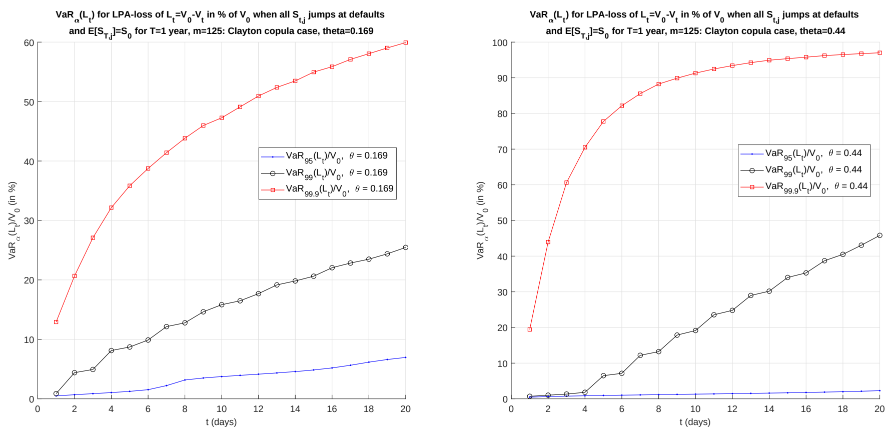

FIGURE 15. The time evolution of Value-at-Risk (in % of V_0) computed with the LPA formula in Theorem 5.2 for $t = 1, 2, \dots, 20$ days of a homogeneous portfolio with $J = 150$ stocks, which has jumps in all stock prices at default times driven by a Clayton copula model with parameters as in Table 6 where $m = 125$. In both panels, condition (7.1.4) holds, i.e. $\mathbb{E}[S_{T,j}] = S_0$. **Left panel:** For $\theta = 0.169$. **Right panel:** For $\theta = 0.44$.

The curves for $\alpha = 95\%$ and 99% VaR plots in Figure 15 show a non-smooth pattern which follows from the same arguments given in Subsection 8.3 regarding the plots in the left and right panels of Figure 9 compared with the middle and right panels in Figure 19 on p.66. Hence, the non-smoothness in the left and right panels of Figure 15 follows the same time pattern as in the left and right panels of Figure 17, which displays the time evolution of Value-at-Risk for the default counting process $N_t^{(m)}$ driven by a Clayton copula model with parameters as in Table 6 where $m = 125$.

Next, in the two panels in Figure 16 we display the time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \dots, 20$ days between a stock portfolio with jumps as in Figure 15 using the LPA-formula (5.5) in Theorem 5.2 and the standard Black-Scholes case, i.e. without jumps, given by in the right panel of Figure 11 computed with the Black-Scholes LPA-formula in Equation (5.30) with parameters as in Table 6. As can be seen in Figure 16, the differences between the jump vs non-jump VaR-cases are huge. For example, the 99.9% VaR for $\theta = 0.44$ in the right panel is for some time points around 4000% higher than the corresponding 99.9% VaR values in the Black-Scholes portfolio case. In our VaR computations in Figure 16, we use the same truncation techniques for the LPA portfolio loss distributions as discussed in Subsection 7.3.

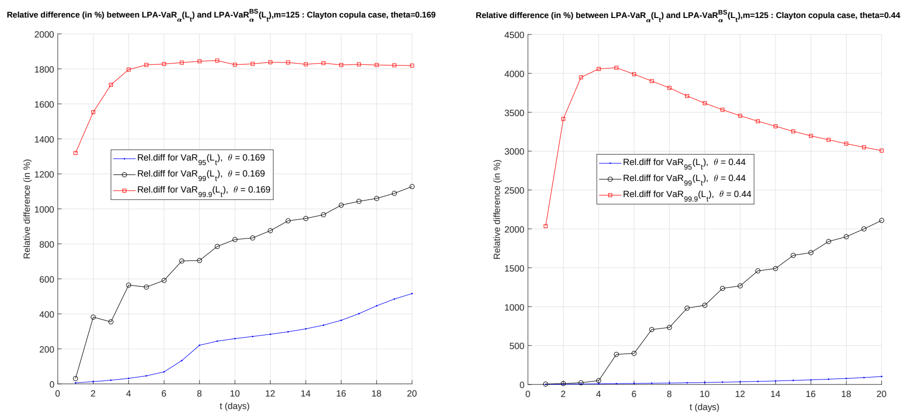

FIGURE 16. The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between a stock portfolio with jumps as in Figure 15 using the LPA formula in Theorem 5.2 and the standard Black-Scholes case, i.e. without jumps, given by right panel in Figure 11. The relative difference is measured with respect to the Black-Scholes case. All parameters for the jump-model are as in Figure 15. **Left panel:** For $\theta = 0.169$ and $\eta = 21.67$. **Right panel:** For $\theta = 0.44$ and $\eta = 13.48$.

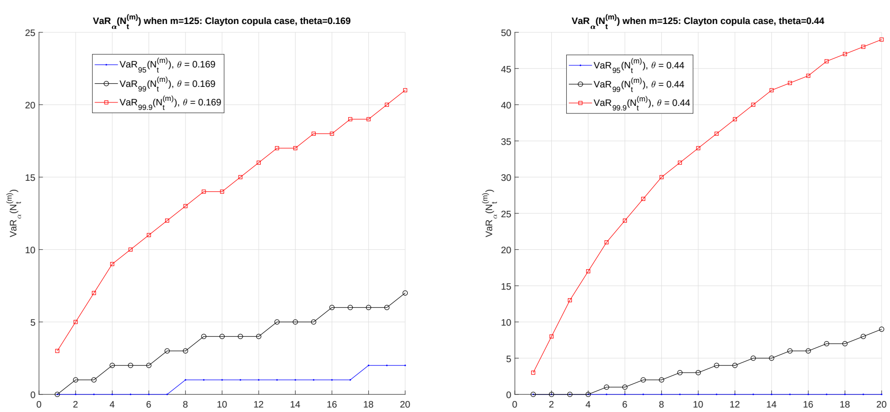

FIGURE 17. The time evolution of Value-at-Risk at $t = 1, 2, \ldots, 20$ days for the default counting process $N_t^{(m)}$ driven by a Clayton copula model with parameters as in Table 6 where $m = 125$. **Left panel:** For $\theta = 0.169$. **Right panel:** For $\theta = 0.44$.

Finally, we can use Figure 16 together with Figure 10 to compare the VaR in stock portfolios with jumps at defaults coming from a Clayton copula compared to corresponding VaR values when jumps are generated by defaults coming from a one-factor Gaussian copula model. For example, from the left panel in Figure 10 we see that, in a Gaussian copula model with $\rho = 0.3$, the relative difference will never exceed 1300% of the 99.9% VaR values in the Black-Scholes portfolio case in the first 20 trading days, while for the Clayton copula with $\theta = 0.169$ (which has the same one-year default correlation as the Gaussian copula model with $\rho = 0.3$), we see from the left panel in Figure 16 that the relative difference will constantly exceed 1800% of the 99.9% VaR values in the Black-Scholes portfolio from the fourth trading day up to at least trading day 20. The differences are even higher when we compare the Gaussian copula case with $\rho = 0.6$ vs. the Clayton copula model with $\theta = 0.44$, see the right panel of Figure 10 compared with right panel in Figure 16.

10. Numerical comparison against Kou model with only negative jumps.

The numerical studies in Sections 7, 8, and 9 considered VaR-values for stock portfolios in the case where the jumps in the stock prices were triggered by defaults from an exogenous group of m entities $\mathbf{C}_1, \ldots, \mathbf{C}_m$ for different credit portfolio default models. Then, in Sections 7, 8, and 9 we also compared these jump-at-default stock portfolio VaR values with corresponding VaR metrics coming from an equity model without jumps, that is, a Black-Scholes portfolio setting under the real probability measure. However, as mentioned in Section 6, comparing our jump-at-default stock price model with only a non-jump stock model will in our view not be fully fair. We believe it is equally important to compare our stock price model containing jumps at defaults with other equity models that includes jumps in the stock price driven by, e.g., a Poisson process. Therefore, in this section we will compute VaR for a stock portfolio model derived from the Kou model, [32], restricted to only having negative jumps, as outlined in Section 6, and then compare these VaR-values with the corresponding VaR-metrics coming from our jump-at-defaults model for a one-factor Gaussian copula model as outlined in Section 5. In this section, we only focus on homogeneous stock portfolios, just as in Subsection 8.3 for the one-factor Gaussian copula model.

First, in Subsection 10.1 we briefly discus how we choose the parameters in the restricted version of the Kou model (6.1) used for our VaR-studies. Then, in Subsection 10.2 we consider a large portfolio with $J = 150$ stocks and use the LPA (large portfolio approximation) formula (6.5) given in Corollary 6.1 for the Kou model (6.1) to compute VaR for this equity portfolio with parameters as in Subsection 10.1. We also compare the stock portfolio VaR-values in the Kou model with the corresponding VaR-metrics in the stock price model with jumps at defaults generated by a one-factor Gaussian copula model.

10.1. The parameters in the Kou model with only negative jumps.

We will consider a homogeneous stock portfolio so that condition (4.34) is satisfied, that is, $S_{0,j} = S_0$, $\mu_j = \mu$, $\sigma_j = \mu$, and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1, \ldots, \mathbf{A}_J$, which implies that the stock prices $S_{t,1}^{(\kappa)}, S_{t,2}^{(\kappa)}, \ldots, S_{t,J}^{(\kappa)}$ in the Kou model (6.1) are exchangeable. Furthermore, the numerical values of $S_{0,j}^{(\kappa)} = S_0$, the drift μ, volatility σ, and "stock-correlation" ρ_S are chosen to be the same as in one-factor Gaussian copula model used in Subsection 8.2, and are given Table 4, see also in Table 8 below. So, what is left to determine is the numerical value of the parameters λ_κ and η_κ in the restricted version of the Kou model $S_{t,j}^{(\kappa)}$ given by (6.1). Here we remind that λ_κ is the rate for the Poisson process driving the jumps in $S_{t,j}^{(\kappa)}$, and η_κ is the parameter for the downward exponentially distributed jump-size in $S_{t,j}^{(\kappa)}$ as stated by (6.1). To find λ_κ and η_κ, we will use the method outlined in Section 6 and given by (6.6) or

(7.1.4), that is, $\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0 = \mathbb{E}\left[S_{T,j}\right]$ and $\mathbb{E}\left[N_{\tilde{T}}\right] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right]$ for two arbitrary fixed time points T and $\tilde{T}$, but where here we let $T = \tilde{T} = 1$ year. Recall that the condition $\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0$ implies, just as in (2.32), that the downward jumps in the Kou model $S_t^{(\kappa)}$ at the jump times of the Poisson process N_t "wipe" out the expected log-growth for a corresponding Black-Scholes model with drift μ up to time T, see also Equation (6.4). Furthermore, we observe that $\mathbb{E}\left[N_{\tilde{T}}\right] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right]$ means that the expected number of jumps by the point processes N_t and $N_t^{(m)}$ will be the same up to time $\tilde{T}$, which for $N_t^{(m)}$ is the same as saying that the expected number of defaults in the group $\mathbf{C}_1, \ldots, \mathbf{C}_m$ up to time $\tilde{T}$ will be given by $\mathbb{E}\left[N_{\tilde{T}}\right] = \lambda_\kappa \tilde{T}$. As discussed in Section 6, if the default times $\tau_1, \tau_2 \ldots, \tau_m$ for $\mathbf{C}_1, \ldots, \mathbf{C}_m$ are exchangeable with default distribution $F(t) = \mathbb{P}\left[\tau_i \leq t\right]$, then the condition $\mathbb{E}\left[N_{\tilde{T}}\right] = \mathbb{E}\left[N_{\tilde{T}}^{(m)}\right]$ can be reformulated as $\lambda_\kappa \tilde{T} = mF(\tilde{T})$. Furthermore, if the exchangeable default times $\tau_1, \tau_2 \ldots, \tau_m$ have constant default intensity λ so that $F(t) = \mathbb{P}\left[\tau_i \leq t\right] = 1 - e^{-\lambda t}$, then Equations (6.6) - (6.10) in Section 6 together with $\tilde{T} = 1$ year renders that λ_κ and η_κ are given by (see also (6.10))

$$\lambda_\kappa = m\left(1 - e^{-\lambda}\right) \qquad \text{and} \qquad \eta_\kappa = \frac{\lambda_\kappa}{\mu} - 1\,. \tag{10.1.1}$$

Note that the calibration of λ_κ and η_κ in (10.1.1) will not involve any of the dependence parameters describing the default times in the stock price model with jumps at defaults generated by $N_t^{(m)}$. Hence, the condition (10.1.1) can be used for any exchangeable factor copula model or intensity based model that will be compared with the Kou model (6.1). In this section, we will only use the one-factor Gaussian copula model in our comparison with the Kou model (6.1), and we will use two different values of the default correlation parameter ρ, namely $\rho = 0.3$ and $\rho = 0.6$, just as in Subsection 8.3, see Table 8. The stock-jump parameter η in the Gaussian copula case are same as in Subsection 8.3, that is, $\eta = 21.98$ for $\rho = 0.3$, and $\eta = 13.92$ when $\rho = 0.6$, see also Subsection 8.3.

TABLE 8. The parameters and related quantities for the one-factor Gaussian copula model and the stock prices $S_{t,j}^{(\kappa)}$ in Kou model (6.1) with only negative jumps calibrated via (10.1.1).

Gauss copula	$m = 125$	$\rho = 0.3, \rho = 0.6$	$F(t) = 1 - e^{-\lambda t}$	$\lambda = 0.0335$	$\mathbb{P}\left[\tau_i \leq 1\right] = 0.0329$
Kou model $S_{t,j}^{(\kappa)}$	$S_0 = 50$	$\mu = 0.15$	$\sigma = 0.2 \quad \rho_S = 0.25$	$\lambda_\kappa = 4.1125$	$\eta_\kappa = 26.4167$

10.2. **VaR over a 20-day period for a large homogeneous stock portfolio where jumps in stocks are due to Kou model with only negative jumps.** In this subsection, we study Value-at-Risk for a large homogeneous portfolio of stocks as a function of time over a 20-day period in time steps of one trading day where the stock prices in the portfolio are modeled as in Equation (6.1) in Section 6, which is similar to the Kou model, [32], restricted to only having negative jumps. Thus, the stock prices in $S_{t,j}^{(\kappa)}$ in (6.1) will be the same as in [32], but here with only negative jumps and where we extend [32] to a portfolio setting so that $S_{t,j}^{(\kappa)}$ are correlated via a Brownian motion (a factor process) $W_{t,0}$ and a Poisson process

N_t that is the same for all stocks in the portfolio and with parameters as in Table 8. In the modified Kou model (6.1), we then study VaR for a portfolio of $J = 150$ stocks by using the LPA approximation formula in Corollary 6.1 and also compare these VaR values with corresponding VaR values in a portfolio with stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model, that is, Definition 4.1 where $N_t^{(m)}$ is the counting process for a one-factor Gaussian copula model with m entities (see also in Subsection 8.3).

The left panel in Figure 18 displays the time evolution of Value-at-Risk (in % of V_0) computed with the LPA formula in Corollary 6.1 for $t = 1, 2, \ldots, 20$ days of a homogeneous portfolio with $J = 150$ stocks in the Kou model (6.1) with only negative jumps and parameters as in Table 8. The middle and right panels in Figure 18 display the time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between a stock portfolio modeled by the restricted Kou model (6.1) given in the left panel and a homogeneous portfolio with $J = 150$ stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4 with default-correlation parameter $\rho = 30\%$ (middle panel) and $\rho = 60\%$ (right panel). The relative difference in Figure 18 is measured with respect to the Kou model (6.1), and the VaR curves of the one-factor Gaussian copula model are displayed in Figure 9. Thus, the middle and right panels in Figure 18 are obtained by taking the difference of the graphs in the left panel of Figure 18 and the corresponding graphs in the left and right plots in Figure 9 divided by the graphs in left panel of Figure 18 and then multiplied by 100 to express the relative difference in units of percent. Recall from Subsection 10.1 that the parameters in Figure 18 are chosen so that $\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0 = \mathbb{E}\left[S_{T,j}\right]$ and $\mathbb{E}\left[N_T\right] = \mathbb{E}\left[N_T^{(m)}\right]$ for $T = 1$ year, and this choice of the parameters in the restricted Kou model (6.1) will hopefully make the comparison with the jump-at-default model in Definition 4.1 where $N_t^{(m)}$ is driven by a one-factor Gaussian copula model somewhat financially "fair".

From the middle and right panels in Figure 18, we clearly see that the stock portfolio VaR values coming from the one-factor Gaussian copula model are much higher for the 99% and 99.9% cases the first 20 days compared to the corresponding VaR numbers in the restricted Kou portfolio model (6.1), even though all the portfolios have the same expected values for the individual stock prices after one year, and the same expected number of jumps after one year. For example, in the right panel in Figure 18, we see that after 9 trading days, the 99.9% VaR coming from a jump-at-default stock model driven by a one-factor Gaussian copula model with $\rho = 60\%$ will be 830% bigger compared with the corresponding Kou portfolio model (6.1) on the same day. Similarly, in the in the middle panel of Figure 18, we see that after 20 trading days the 99% VaR coming in the form of a jump-at-default stock model driven by a one-factor Gaussian copula model with $\rho = 30\%$ will be 302% bigger compared with the corresponding Kou portfolio model (6.1) on the same day.

As seen in Figure 19, the main reason for the significantly bigger stock portfolio VaR values in the jump-at-default equity model in Definition 4.1 where $N_t^{(m)}$ is driven by a one-factor Gaussian copula compared to the Kou portfolio model (6.1) with jumps at a Poisson process N_t are due to the much larger tail behavior of $N_t^{(m)}$ compared with the tail characteristics of N_t even though $\mathbb{E}\left[N_T\right] = \mathbb{E}\left[N_T^{(m)}\right]$

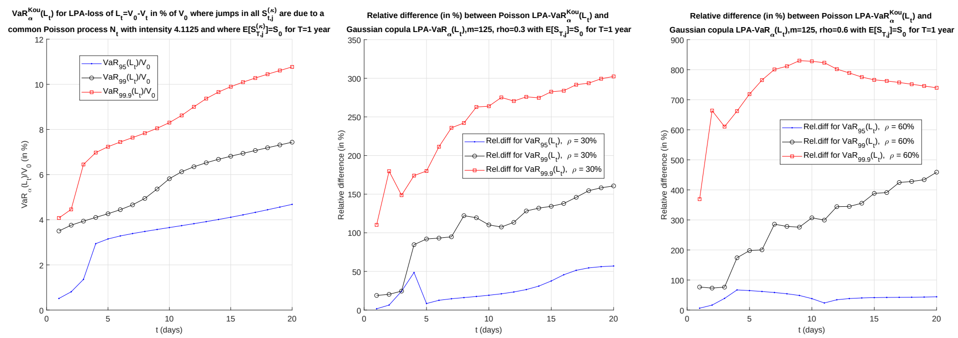

FIGURE 18. **Left panel:** The time evolution of Value-at-Risk (in % of V_0) computed with the LPA formula in Corollary 6.1 for $t = 1, 2, \ldots, 20$ days of a homogeneous portfolio with $J = 150$ stocks in the Kou model (6.1) with only negative jumps and parameters as in Table 8. **Middle and right panel:** The time evolution of the relative difference of Value-at-Risk (in %) for $t = 1, 2, \ldots, 20$ days between a stock portfolio with jumps as in the Kou model in the left panel and Figure 9, which displays VaR for a portfolio with stocks which has jumps in all stock prices at default times driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4 with default-correlation parameter $\rho = 30\%$ (middle panel) and $\rho = 60\%$ (right panel). The relative difference is measured with respect to the Kou model (6.1). In all panels, it holds that $\mathbb{E}\left[S_{T,j}^{(\kappa)}\right] = S_0 = \mathbb{E}\left[S_{T,j}\right]$ and $\mathbb{E}\left[N_T\right] = \mathbb{E}\left[N_T^{(m)}\right]$ for $T = 1$ year.

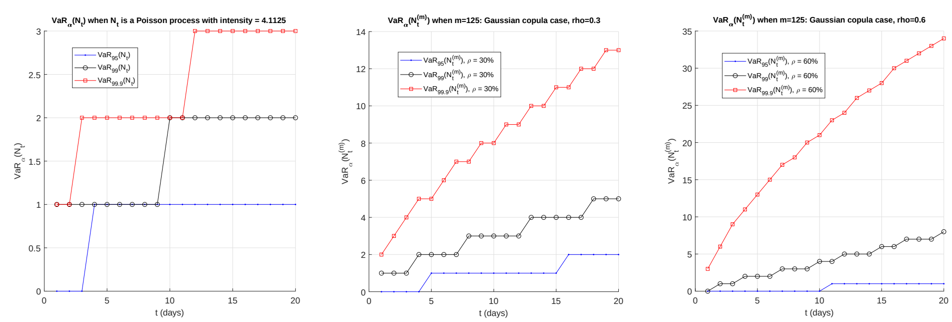

FIGURE 19. The time evolution of Value-at-Risk at $t = 1, 2, \ldots, 20$ days for Poisson process N_t with intensity 4.113 (**left panel**) and default counting process $N_t^{(m)}$ driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4 and default-correlation parameter $\rho = 30\%$ (**middle panel**) and $\rho = 60\%$ (**right panel**).

for $T = 1$ year. The left panel in Figure 19 displays the time evolution of Value-at-Risk at $t = 1, 2, \ldots, 20$ days for a Poisson process N_t with intensity 4.113 (so that $\mathbb{E}\left[N_T\right] = \mathbb{E}\left[N_T^{(m)}\right]$ for $T = 1$ year), while the middle and right panels in Figure

19 plot the corresponding VaR values coming from default counting process $N_t^{(m)}$ driven by a one-factor Gaussian copula model with $m = 125$ and parameters as in Table 4 and default-correlation parameter $\rho = 30\%$ (middle panel) and $\rho = 60\%$ (right panel). In Figure 19, we see that $\mathrm{VaR}_{99.9\%}(N_t)$ for the Poisson process N_t will never exceed 3 jumps the first 20 trading days, while for $N_t^{(m)}$ driven by the one-factor Gaussian copula model, the $\mathrm{VaR}_{99.9\%}\left(N_t^{(m)}\right)$ will steadily increase up to 13 jumps, i.e. 13 defaults, the first 20 trading days when $\rho = 30\%$ and up to 34 jumps, i.e. 34 defaults, when $\rho = 60\%$ during the same period. These are huge differences compared to the Poisson model N_t, even though it holds that $\mathbb{E}[N_T] = \mathbb{E}\left[N_T^{(m)}\right]$ for $T = 1$ year and is also explaining the large differences for the corresponding stock portfolio VaR values displayed in the middle and right panels of Figure 18. Of course, the differences in the middle and right panels in Figure 18 will be even higher if ρ is increased, or if we use a Clayton copula for the default times that create $N_t^{(m)}$ with, e.g., parameters $\theta = 0.169$ or $\theta = 0.44$, as seen in Section 9.

Appendix A. Proof of Proposition 2.6, Theorem 2.14, Corollary 4.11 and Corollary 6.1. In this appendix, we first give a proof of Proposition 2.6 in Subsection A.1, and then a proof of Theorem 2.14 in Subsection A.2. A proof of Corollary 4.11 is given in Subsection A.3, and finally Corollary 6.1 is proved in Subsection A.4.

A.1. Proof of Proposition 2.6.

Proof. Let $\mathcal{F}_t^W = \sigma(W_s; s \leq t)$ be the filtration generated by the Brownian motion W_t and let $\mathcal{H}_t^i = \sigma\left(1_{\{\tau_i \leq s\}}; s \leq t\right)$ be the filtration generated by each default time τ_i, and define the sigma-algebra $\mathcal{V}$ as $\mathcal{V} = \sigma\left(\tilde{V}_1, \dots, \tilde{V}_m\right)$. Next, we define the full filtration $\mathcal{F}_t$ as

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \bigvee_{i=1}^{m} \mathcal{H}_t^i \vee \mathcal{V}\,. \tag{A.1.1}$$

Then, Y_t in Definition 2.1 is a semimartingale with respect to the filtration $\mathcal{F}_t$. To see this, first note that since W_t is a Brownian motion, it is a martingale with respect to its own filtration $\mathcal{F}_t^W$. But, due to Definition 2.1 the process W_t is independent of $\tau_1, \tau_2 \dots, \tau_m$ and $\tilde{V}_1, \dots, \tilde{V}_m$, so W_t will also be a martingale with respect to the full filtration $\mathcal{F}_t$ given by (A.1.1). Hence, from (2.2) we see that Y_t can be written as a sum of local martingale with respect to $\mathcal{F}_t$, that is, σW_t and a finite variation process, i.e. $\mu t + \sum_{i=1}^{m} \tilde{V}_i 1_{\{\tau_i \leq t\}}$ since $\tilde{V}_i$ have bounded expected values. From Theorem 1 on p.102 in [45], we therefore conclude that Y_t is a semimartingale with respect to the filtration $\mathcal{F}_t$ defined as in (A.1.1). Next, we note that the differential form (2.1) can be rewritten as

$$S_t = S_0 + \int_0^t S_{s-} dY_s \tag{A.1.2}$$

and letting S_t be given by $S_t = S_0 \tilde{S}_t$, then (A.1.2) can be rewritten as

$$S_0 \tilde{S}_t = S_0 \left(1 + \int_0^t \tilde{S}_{s-} dY_s\right)$$

that is,

$$\tilde{S}_t = 1 + \int_0^t \tilde{S}_{s-} dY_s \,. \tag{A.1.3}$$

Hence, if we can find a solution to $\tilde{S}_t$ in the SDE (A.1.3), then a solution to S_t in (A.1.2) is obtained from the relation $S_t = S_0 \tilde{S}_t$. Thus, for notational convenience we will without loss of generality assume that $S_0 = 1$ and drop the tilde notation in (A.1.3) so that we have

$$S_t = 1 + \int_0^t S_{s-} dY_s \,. \tag{A.1.4}$$

Next, from Theorem 37 on p.84 in [45], we conclude that S_t in (A.1.4) is a semi-martingale given by

$$S_t = \exp\left(Y_t - \frac{1}{2}[Y,Y]_t^c\right) \prod_{0<s\leq t} (1+\Delta Y_s) \exp(-\Delta Y_s) \tag{A.1.5}$$

where, as usual, $[Y,Y]_t^c$ denotes the path-by-path continuous part of the quadratic variation $[Y,Y]_t$, see p.70 in [45]. Since $\mu t + \sigma W_t$ is a continuous process, we have

$$Y_t = \mu t + \sigma W_t + \sum_{i=1}^m \tilde{V}_i 1_{\{\tau_i \leq t\}} = \mu t + \sigma W_t + \sum_{0<s\leq t} \Delta Y_s \tag{A.1.6}$$

so that

$$\exp\left(Y_t - \frac{1}{2}[Y,Y]_t^c\right) = \exp\left(\mu t + \sigma W_t - \frac{1}{2}[Y,Y]_t^c + \sum_{0<s\leq t} \Delta Y_s\right) \,. \tag{A.1.7}$$

Furthermore, note that

$$\prod_{0<s\leq t} (1+\Delta Y_s) \exp(-\Delta Y_s) = \exp\left(-\sum_{0<s\leq t} \Delta Y_s\right) \prod_{0<s\leq t} (1+\Delta Y_s) \tag{A.1.8}$$

and from the definition of a Brownian motion and since $[Y,Y]_t^c$ is the continuous part of the quadratic variation $[Y,Y]_t$, we get

$$[Y,Y]_t^c = \sigma^2 t \,. \tag{A.1.9}$$

So, (A.1.7)-(A.1.9) in (A.1.5) then gives

$$S_t = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \prod_{0<s\leq t} (1+\Delta Y_s) \tag{A.1.10}$$

and in view of (2.2) we have

$$\prod_{0<s\leq t} (1+\Delta Y_s) = \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}}\right)$$

so that (A.1.10) can be rewritten as

$$S_t = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \prod_{i=1}^m \left(1 + \tilde{V}_i 1_{\{\tau_i \leq t\}}\right) \,. \tag{A.1.11}$$

Recall that we set $S_0 = 1$, but from the arguments leading to (A.1.3), we can let S_0 be an arbitrary positive constant, and using this in (A.1.11) finally gives

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)\prod_{i=1}^{m}\left(1 + \tilde{V}_i 1_{\{\tau_i \le t\}}\right)$$

which proves (2.3), and this concludes the proposition. □

A.2. **Proof of Theorem 2.14.**

Proof. We start with $\mathbb{P}[S_t \le x]$ and note that

$$\mathbb{P}[S_t \le x] = \sum_{k=0}^{m}\mathbb{P}\left[S_t \le x \,|\, N_t^{(m)} = k\right]\mathbb{P}\left[N_t^{(m)} = k\right] \tag{A.2.1}$$

where Corollary 2.13 implies that

$$\begin{aligned}&\mathbb{P}\left[S_t \le x \,|\, N_t^{(m)} = k\right]\\ &= \mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{k} U_n\right)\middle|\, N_t^{(m)} = k\right].\end{aligned} \tag{A.2.2}$$

From Definition 2.1, we know that W_t is independent of the default times $\tau_1, \tau_2 \dots,$ τ_m, and from Definition 2.10, we also know that the sequence $U_1, \dots, U_m$ is independent of $\tau_1, \tau_2 \dots, \tau_m$. Thus, the process $N_t^{(m)}$ is independent of both W_t and $U_1, \dots, U_m$, which in (A.2.2) gives

$$\begin{aligned}&\mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{k} U_n\right) \le x \,\middle|\, N_t^{(m)} = k\right]\\ &= \mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{k} U_n\right) \le x\right]\end{aligned} \tag{A.2.3}$$

and the right-hand side of (A.2.3) can be simplified to

$$\begin{aligned}&\mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \sum_{n=1}^{k} U_n\right) \le x\right]\\ &= \mathbb{P}\left[\sigma W_t - \sum_{n=1}^{k} U_n \le \ln\frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right].\end{aligned} \tag{A.2.4}$$

Let X and G_k be independent random variables, where X is a standard normal random variable and G_k is a gamma-distributed random variable so that $G_k \overset{d}{=} \text{Gamma}(k, \eta)$ where, $k \ge 1$ is an integer. Then, we note that

$$\sigma W_t \overset{d}{=} \sigma\sqrt{t}X \quad \text{and} \quad \sum_{n=1}^{k} U_n \overset{d}{=} G_k \overset{d}{=} \text{Gamma}(k, \eta) \tag{A.2.5}$$

where the last equality follows from the fact that a sum of k independent exponentially distributed random variables all with parameters η has distribution $\text{Gamma}(k, \eta)$. From Definition 2.10, we know that $U_1, \dots, U_m$ are independent of W_t, which motivates why X and G_k in (A.2.5) are independent random variables.

Next, let U and V be independent random variables with distributions $F_U(u)$ and $F_V(v)$. From standard probability theory, we know that

$$\mathbb{P}\left[U+V\leq z\right]=\int F_U(z-v)dF_V(v) \tag{A.2.6}$$

see Theorem 2.1.1 on p.47 in [15]. If we define U and V as

$$U=\sigma\sqrt{t}X \quad \text{and} \quad V=-G_k \tag{A.2.7}$$

where X and G_k are the same as in (A.2.5), then we have that

$$F_U(u)=\Phi\left(\frac{u}{\sigma\sqrt{t}}\right) \quad \text{and} \quad F_V(v)=1-F_{G_k}(-v) \quad \text{where} \quad v\in(-\infty,0] \tag{A.2.8}$$

so that

$$dF_V(v)=f_{G_k}(-v)dv \quad \text{for} \quad v\in(-\infty,0] \tag{A.2.9}$$

where $F_{G_k}(x)$ and $f_{G_k}(x)$ are the distribution function and density function to $G_k \stackrel{d}{=} \text{Gamma}(k,\eta)$, and as usual $\Phi(x)$ is the distribution function to a standard normal random variable. Now, (A.2.7), (A.2.8) and (A.2.9), in (A.2.6) then renders as

$$\mathbb{P}\left[\sigma\sqrt{t}X-G_k\leq z\right]=\int_{-\infty}^{0}\Phi\left(\frac{z-v}{\sigma\sqrt{t}}\right)f_{G_k}(-v)dv \tag{A.2.10}$$

and by making the change of variables $y=-v$ in (A.2.10), the integral on the right-hand side of (A.2.10) can be rewritten as

$$\mathbb{P}\left[\sigma\sqrt{t}X-G_k\leq z\right]=\int_{0}^{\infty}\Phi\left(\frac{z+y}{\sigma\sqrt{t}}\right)f_{G_k}(y)dy\,. \tag{A.2.11}$$

By letting $z=\ln\frac{x}{S_0}-\left(\mu-\frac{1}{2}\sigma^2\right)$ and $f_{G_k}(y)=\frac{\eta e^{-\eta y}(\eta y)^{k-1}}{(k-1)!}$ in (A.2.11), together with the relation (A.2.5), we get for any integer $k\geq 1$ that the right-hand side of (A.2.4) can be written as

$$\begin{aligned}&\mathbb{P}\left[\sigma W_t-\sum_{n=1}^{k}U_n\leq\ln\frac{x}{S_0}-\left(\mu-\frac{1}{2}\sigma^2\right)t\right]\\&=\int_0^\infty\Phi\left(\frac{\ln\frac{x}{S_0}-\left(\mu-\frac{1}{2}\sigma^2\right)t+y}{\sigma\sqrt{t}}\right)\frac{\eta e^{-\eta y}\left(\eta y\right)^{k-1}}{(k-1)!}\,dy\,.\end{aligned} \tag{A.2.12}$$

By combining (A.2.3) and (A.2.4) with (A.2.12), we get for for any integer $k\geq 1$ that

$$\begin{aligned}&\mathbb{P}\left[S_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t-\sum_{n=1}^{k}U_n\right)\leq x\,\middle|\,N_t^{(m)}=k\right]\\&=\Psi_k\left(x,t,\mu,\sigma,S_0,\eta\right)\end{aligned} \tag{A.2.13}$$

where the mappings $\Psi_k\left(x,t,\mu,\sigma,u,\eta\right)$ for $u>0$ are defined as

$$\begin{aligned}&\Psi_k\left(x,t,\mu,\sigma,u,\eta\right)\\&=\int_0^\infty\Phi\left(\frac{\ln\frac{x}{u}-\left(\mu-\frac{1}{2}\sigma^2\right)t+y}{\sigma\sqrt{t}}\right)\frac{\eta e^{-\eta y}\left(\eta y\right)^{k-1}}{(k-1)!}\,dy \quad \text{for integers } k\geq 1\,.\end{aligned} \tag{A.2.14}$$

When $k=0$, we have no defaults, and thus there are no jump-terms in the exponential expression of (A.2.3), implying that (A.2.3) reduces to

$$\mathbb{P}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \leq x \,\middle|\, N_t^{(m)} = 0\right] = \Psi_0\left(x, t, \mu, \sigma, S_0, \eta\right) \quad \text{(A.2.15)}$$

where $\Psi_0(x, t, \mu, \sigma, u, \eta)$ for $u > 0$ is defined as

$$\begin{aligned}\Psi_0\left(x, t, \mu, \sigma, u, \eta\right) &= \mathbb{P}\left[u \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \leq x\right] \\ &= \Phi\left(\frac{\ln\frac{x}{u} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right). \end{aligned} \quad \text{(A.2.16)}$$

Thus, (A.2.13) - (A.2.16) together with (A.2.1) and (A.2.2) implies that

$$\mathbb{P}\left[S_t \leq x\right] = \sum_{k=0}^{m} \Psi_k\left(x, t, \mu, \sigma, S_0, \eta\right)\mathbb{P}\left[N_t^{(m)} = k\right]$$

which proves (2.18), (2.19), and (2.20). Next, consider the loss distribution $F_{L_t^{(S)}}(x) = \mathbb{P}\left[L_t^{(S)} \leq x\right]$. From the definition of $L_t^{(S)}$ in (4.9), after some trivial computations, we get that

$$F_{L_t^{(S)}}(x) = \mathbb{P}\left[L_t^{(S)} \leq x\right] = 1 - \mathbb{P}\left[\frac{S_t}{S_0} \leq 1 - \frac{x}{S_0}\right] \quad \text{(A.2.17)}$$

and we can therefore reuse the formula for $\mathbb{P}[S_t \leq x]$ in (2.18) by letting $S_0 = 1$ in (2.18), and replace x in (2.18) with $1 - \frac{x}{S_0}$, rendering that

$$\mathbb{P}\left[L_t^{(S)} \leq x\right] = 1 - \sum_{k=0}^{m} \Psi_k\left(1 - \frac{x}{S_0}, t, \mu, \sigma, 1, \eta\right)\mathbb{P}\left[N_t^{(m)} = k\right]$$

which proves (2.18). Next, we prove the expressions for the density $f_{S_t}(x)$ to S_t and first note that $f_{S_t}(x) = \frac{d}{dx}\mathbb{P}[S_t \leq x]$, so (2.18) then implies that

$$f_{S_t}(x) = \sum_{k=0}^{m} \frac{\partial}{\partial x}\Psi_k\left(x, t, \mu, \sigma, S_0, \eta\right)\mathbb{P}\left[N_t^{(m)} = k\right]. \quad \text{(A.2.18)}$$

Next, we define $\psi_k(x, t, \mu, \sigma, S_0, \eta)$ as

$$\psi_k\left(x, t, \mu, \sigma, S_0, \eta\right) = \frac{\partial}{\partial x}\Psi_k\left(x, t, \mu, \sigma, S_0, \eta\right) \quad \text{(A.2.19)}$$

and for $k \geq 1$ with $x > 0, t > 0$, we then get from (A.2.14) and (A.2.19) and some elementary computations that

$$\begin{aligned}&\psi_k\left(x, t, \mu, \sigma, S_0, \eta\right) \\ &= \frac{1}{x\sigma\sqrt{t}}\int_0^\infty \varphi\left(\frac{\ln\frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t + y}{\sigma\sqrt{t}}\right)\frac{\eta e^{-\eta y}\left(\eta y\right)^{k-1}}{(k-1)!}\,dy \quad \text{for } 0 < k \leq m\end{aligned} \quad \text{(A.2.20)}$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the density of a standard normal random variable. In the same way, (A.2.16) and (A.2.19) imply that $\psi_0(x, t, \mu, \sigma, S_0, \eta)$ for $S_0 > 0, x > 0$, and $t > 0$ is given by

$$\psi_0\left(x, t, \mu, \sigma, S_0, \eta\right) = \frac{1}{x\sigma\sqrt{t}}\varphi\left(\frac{\ln\frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right). \quad \text{(A.2.21)}$$

Hence, (A.2.19) with (A.2.20)-(A.2.21) inserted into (A.2.18) proves (2.22) - (2.24). Finally, we note that

$$\mathbb{E}\left[S_t\right] = \sum_{k=0}^{m} \mathbb{E}\left[S_t \,|\, N_t^{(m)} = k\right] \mathbb{P}\left[N_t^{(m)} = k\right] \tag{A.2.22}$$

where Corollary 2.13 implies that

$$\mathbb{E}\left[S_t \,|\, N_t^{(m)} = k\right] = \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t - \sum_{n=1}^{k} U_n\right) \middle| N_t^{(m)} = k\right]. \tag{A.2.23}$$

By using exactly the same arguments which led to the right-hand side in (A.2.3), we have that

$$\begin{aligned} &\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t - \sum_{n=1}^{k} U_n\right) \middle| N_t^{(m)} = k\right] \\ &= \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t - \sum_{n=1}^{k} U_n\right)\right]. \end{aligned} \tag{A.2.24}$$

Furthermore, since W_t are independent of the jump terms $U_1, \ldots, U_m$ we get that the right-hand side of (A.2.23) can be rewritten as

$$\begin{aligned} &\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t - \sum_{n=1}^{k} U_n\right)\right] \\ &= \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)\right] \mathbb{E}\left[e^{-\sum_{n=1}^{k} U_n}\right]. \end{aligned} \tag{A.2.25}$$

From standard Black-Scholes theory, we have

$$\mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)\right] = S_0 e^{\mu t} \tag{A.2.26}$$

and from (A.2.5), we also note that

$$\mathbb{E}\left[e^{-\sum_{n=1}^{k} U_n}\right] = \mathbb{E}\left[e^{-G_K}\right] = \mathcal{L}_{G_K}(1) = \left(\frac{\eta}{\eta+1}\right)^k \tag{A.2.27}$$

where $\mathcal{L}_{G_K}(s)$ is the Laplace transform to $G_k \overset{d}{=} \text{Gamma}(k, \eta)$ with $k \geq 1$ and $\eta > 0$ obtained from the moment generating function $\mathcal{M}_{G_K}(s)$ via the relation $\mathcal{L}_{G_K}(s) = \mathcal{M}_{G_K}(-s)$, and where standard probability theory gives us that $\mathcal{L}_{G_K}(s) = \left(\frac{\eta}{\eta+s}\right)^k$ for $s > -\eta$. Hence, combining (A.2.24)-(A.2.27) and inserting these relations in (A.2.23) for integers $k = 1, 2, \ldots, m$, we get

$$\mathbb{E}\left[S_t \,|\, N_t^{(m)} = k\right] = S_0 e^{\mu t} \left(\frac{\eta}{\eta+1}\right)^k \tag{A.2.28}$$

and since $\left(\frac{\eta}{\eta+1}\right)^0 = 1$, then (A.2.28) will also hold for $k = 0$. Thus, (A.2.28) for $k = 0, 1, \ldots, m$ in (A.2.22) implies that

$$\mathbb{E}\left[S_t\right] = S_0 e^{\mu t} \sum_{k=0}^{m} \left(\frac{\eta}{\eta+1}\right)^k \mathbb{P}\left[N_t^{(m)} = k\right] = S_0 e^{\mu t} \mathbb{E}\left[\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\right]$$

which proves (2.17). Finally, by using (A.2.28), we have

$$\begin{aligned}\mathbb{E}\left[S_t \mid N_t^{(m)}\right] &= \sum_{k=0}^{m} \mathbb{E}\left[S_t \mid N_t^{(m)} = k\right] 1_{\left\{N_t^{(m)}=k\right\}} \\ &= \sum_{k=0}^{m} S_0 e^{\mu t}\left(\frac{\eta}{\eta+1}\right)^k 1_{\left\{N_t^{(m)}=k\right\}} \\ &= S_0 e^{\mu t}\left(\frac{\eta}{\eta+1}\right)^{N_t^{(m)}}\end{aligned}$$

which together with (A.2.28) proves (2.16), and this concludes the theorem. □

A.3. **Proof of Corollary 4.11.**

Proof. From (4.21) in Theorem 4.8, we have

$$F_{L_t^{\Delta V}}(x) = \mathbb{P}\left[L_t^{\Delta V} \le x\right] = 1 - \mathbb{P}\left[\sum_{j=1}^{J} X_{t,j} \le -\frac{x}{S_0}\right] 1 - \mathbb{P}\left[\sum_{j=1}^{J} Z_{t,j} \le -\frac{x}{S_0}\right] \tag{A.3.1}$$

since there are now jumps, where $Z_{t,j}$ is defined as in (4.23) in Theorem 4.8. Now, (4.33) and (4.25) in Theorem 4.8 with some elementary computations together with the standard normal symmetry property $1-\Phi(-y) = \Phi(y)$ then imply that

$$\mathbb{P}\left[L_t^{\Delta V} \le x\right] = \Phi\left(\frac{\frac{x}{S_0} + \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t}{\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}}\right)$$

where $\Phi(x)$ is the distribution function of a standard normal random variable, and this proves (4.40). Furthermore, from the definition in (4.37), we know that $\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right) = F^{-1}_{L_t^{\Delta V}}(\alpha)$, so this with the distribution of $F_{L_t^{\Delta V}}(x)$ in (4.40) will then after some trivial computations yield that

$$\begin{aligned}&\mathrm{VaR}_\alpha\left(L_t^{\Delta V}\right) \\ &= S_0\left(\sqrt{t\left(\left(\sum_{j=1}^{J}\sigma_j\rho_{S,j}\right)^2 + \sum_{j=1}^{J}\sigma_j^2\left(1-\rho_{S,j}^2\right)\right)}\,\Phi^{-1}(\alpha) - \sum_{j=1}^{J}\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t\right)\end{aligned}$$

which proves (4.41). Finally, if we set the portfolio weights to $w_j = 1$ for all companies $\mathbf{A}_1,\ldots,\mathbf{A}_J$, and if their stock prices $S_{t,1}^{(\mathrm{BS})},\ldots,S_{t,J}^{(\mathrm{BS})}$ also satisfy (4.34) in Remark 4.9, we get an equally value-weighted portfolio where $S_{0,j} = S_0, \mu_j = \mu, \sigma_j = \sigma$, and $\rho_{S,j} = \rho_S$ for all firms $\mathbf{A}_1,\ldots,\mathbf{A}_J$, and using this in (4.40)-(4.41) with some computations gives us expressions (4.42)-(4.43), which concludes the corollary. □

A.4. **Proof of Corollary 6.1.**

Proof. First, identical computations to those that lead to (A.2.28) in the proof of Theorem 2.14 gives us

$$\mathbb{E}\left[S_{t,j}^{(\kappa)}\,\Big|\,N_t=k\right]=S_{0,j}\,e^{\mu t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^k. \tag{A.4.1}$$

Furthermore, using (A.4.1) with similar computations as in the proof of Theorem 2.14 then gives us that

$$\begin{aligned}\mathbb{E}\left[S_{t,j}^{(\kappa)}\,\Big|\,N_t\right]&=\sum_{k=0}^{\infty}\mathbb{E}\left[S_{t,j}^{(\kappa)}\,\Big|\,N_t=k\right]1_{\{N_t=k\}}\\&=\sum_{k=0}^{\infty}S_{0,j}e^{\mu_j t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^k 1_{\{N_t=k\}}\\&=S_{0,j}e^{\mu_j t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^{N_t}\end{aligned}$$

which, together with (A.4.1), proves (6.3). Next, note that

$$\mathbb{E}\left[S_{t,j}^{(\kappa)}\right]=\sum_{k=0}^{\infty}\mathbb{E}\left[S_{t,j}^{(\kappa)}\,\Big|\,N_t=k\right]\mathbb{P}\left[N_t=k\right] \tag{A.4.2}$$

so inserting (A.4.1) into (A.4.2) together with $\mathbb{P}\left[N_t=k\right]=e^{-\lambda_\kappa t}\frac{(\lambda_\kappa t)^k}{k!}$ then gives that

$$\begin{aligned}\mathbb{E}\left[S_{t,j}^{(\kappa)}\right]&=\sum_{k=0}^{\infty}S_{0,j}\,e^{\mu t}\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^k e^{-\lambda_\kappa t}\frac{(\lambda_\kappa t)^k}{k!}\\&=S_{0,j}\,e^{(\mu-\lambda_\kappa)t}\sum_{k=0}^{\infty}\left(\frac{\eta_\kappa\lambda_\kappa t}{\eta_\kappa+1}\right)^k\frac{1}{k!}\\&=S_{0,j}\exp\left(\left(\mu-\lambda_\kappa\right)t\right)\exp\left(\frac{\eta_\kappa\lambda_\kappa t}{\eta_\kappa+1}\right)\\&=S_{0,j}\exp\left(\left(\mu-\frac{\lambda_\kappa}{\eta_\kappa+1}\right)t\right)\end{aligned} \tag{A.4.3}$$

which proves (6.4). Next, we prove (6.5), and first note that if the stock portfolio is homogeneous so that condition (4.34) holds, then $S_{0,j}^{(\kappa)}=S_0,\mu_j=\mu,\sigma_j=\sigma$, and $\rho_{S,j}=\rho_S$ for all firms $\mathbf{A}_1,\dots,\mathbf{A}_J$ so that the $S_{t,1}^{(\kappa)},S_{t,2}^{(\kappa)},\dots,S_{t,J}^{(\kappa)}$ are exchangeable. Then, we observe that (5.12) - (5.19) in Theorem 5.2 will also hold if $N_t^{(m)}$ is replaced by a Poisson process N_t, and this fact will for large J therefore yield

$$\begin{aligned}&\mathbb{P}\left[L_t^{V,\kappa}\le x\right]\\&\approx\mathbb{P}\left[JS_0\left(1-\exp\left(\left(\mu-\frac{1}{2}\sigma^2\rho_S^2\right)t+\sigma\rho_S W_{t,0}\right)\left(\frac{\eta_\kappa}{\eta_\kappa+1}\right)^{N_t}\right)\le x\right].\end{aligned} \tag{A.4.4}$$

Using (A.4.4) with similar steps as in (5.20) - (5.22) with $\mathbb{P}\left[N_t = k\right] = e^{-\lambda_\kappa t}\frac{(\lambda_\kappa t)^k}{k!}$ then implies that

$$\mathbb{P}\left[L_t^{V,\kappa} \leq x\right]$$
$$\approx 1 - \sum_{k=0}^{\infty} e^{-\lambda_\kappa t}\frac{(\lambda_\kappa t)^k}{k!}\Phi\left(\frac{\ln\left(\left(1-\frac{x}{JS_0}\right)\left(\frac{\eta+1}{\eta}\right)^k\right) - \left(\mu - \frac{1}{2}\sigma^2\rho_S^2\right)t}{\sigma\rho_S\sqrt{t}}\right) \quad \text{for large } J$$

which proves (6.5), and this concludes the corollary. □

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